GEOMETRIC REALIZATIONS OF WAKIMOTO MODULES AT THE CRITICAL LEVEL

EDWARD FRENKEL AND DENNIS GAITSGORY

ABSTRACT. We study the Wakimoto modules over the affine Kac-Moody algebras at the critical level from the point of view of the equivalences of categories proposed in our previous works, relating categories of representations and certain categories of sheaves. In particular, we describe explicitly geometric realizations of the Wakimoto modules as Hecke eigen-D-modules on the affine Grassmannian and as quasi-coherent sheaves on the flag variety of the Langlands dual group.

INTRODUCTION

Wakimoto modules, introduced in [W, FF1, FF2], have many applications in representation theory of affine Kac-Moody algebras. In our previous papers [FG1]–[FG4] we have undertaken a study of representations of affine Kac-Moody algebras at the critical level in the framework of the local geometric Langlands correspondence. Wakimoto modules play an important role in it. In this paper we elucidate further the geometric meaning of Wakimoto modules from the point of view of the equivalences of categories proposed in [FG2, FG4]. These equivalences relate categories of representations of the affine Kac-Moody algebras at the critical level to certain categories of D-modules and quasi-coherent sheaves.

0.1. The first step in establishing a connection between representation theory of an affine Kac-Moody algebra at the critical level $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ and geometric Langlands correspondence is the description of the center $\mathfrak{Z}_{\mathfrak{g}}$ of the completed universal enveloping algebra of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$. According to a theorem of [FF3, F], $\mathfrak{Z}_{\mathfrak{g}}$ is isomorphic to the algebra of functions on the space $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^{\times})$ of $\check{\mathfrak{g}}$-opers on the punctured disc, where $\check{\mathfrak{g}}$ is the Langlands dual Lie algebra to $\mathfrak{g}$. This isomorphism was proved in [FF3, F] algebraically, in the framework of representation theory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, and in particular using Wakimoto modules. It may be reformulated as an isomorphism

$$(0.1) \qquad\qquad \mathrm{map}_{\mathrm{alg}} : \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \xrightarrow{\ \sim\ } \mathrm{Op}_{\check{\mathfrak{g}},X},$$

defined for an arbitrary smooth algebraic curve X. Here $\mathfrak{z}_{\mathfrak{g},X}$ is the center of the chiral algebra on X associated to $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ and $\mathrm{Op}_{\check{\mathfrak{g}},X}$ is the scheme of jets of $\check{\mathfrak{g}}$-opers on X.

We recall from [BD] (see also [FG2], Sect. 1) that a $\check{\mathfrak{g}}$-oper on X is a $\check{G}$-bundle on X equipped with a reduction to a Borel subgroup $\check{B}$ and a connection satisfying a certain transversality condition. This means that $\mathrm{Op}_{\check{\mathfrak{g}},X}$ carries a tautological $\check{G}$-bundle equipped with a $\check{B}$-reduction and a connection along X. Defining a morphism $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \to \mathrm{Op}_{\check{\mathfrak{g}},X}$ is equivalent to defining the pull-backs of these data to $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X})$.

Can we construct these data on $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X})$ in a natural way? This question was addressed by A. Beilinson and V. Drinfeld in [BD]. They constructed these data using the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ and the geometric Satake correspondence which identifies the category

of $G[[t]]$-equivariant D-modules on Gr_G with the category $\mathrm{Rep}(\check{G})$ of representations of $\check{G}$, see [MV]. Thus, they obtained a map

$$\mathrm{map}_{\mathrm{geom}} : \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \to \mathrm{Op}_{\check{\mathfrak{g}},X} \,.$$

We recall its definition in Sect. 1.5 below. We note that the construction of the map $\mathrm{map}_{\mathrm{geom}}$ in [BD] relies on the existence of the isomorphism (0.1). However, *a priori* it is not clear whether $\mathrm{map}_{\mathrm{geom}}$ coincides with $\mathrm{map}_{\mathrm{alg}}$, nor whether $\mathrm{map}_{\mathrm{geom}}$ is an isomorphism. Beilinson and Drinfeld proved this in [BD] by showing that both of these maps are compatible with actions of certain Lie algebroids and that this property essentially characterizes these maps uniquely.

In Sects. 1–2 of this paper we give a different proof of the fact that the maps $\mathrm{map}_{\mathrm{alg}}$ and $\mathrm{map}_{\mathrm{geom}}$ coincide. This proof uses Wakimoto modules in an essential way, in particular, their behavior under the "Harish-Chandra convolution" functors which was described in [FG2]. Using this additional structure, we will see the emergence in representation theoretic context not only of the geometric data of opers mentioned above (the "birth of opers", as Beilinson and Drinfeld had put it), but also the geometric data of Miura opers which parametrize Wakimoto modules.

We remark that our proof of the coincidence of $\mathrm{map}_{\mathrm{alg}}$ and $\mathrm{map}_{\mathrm{geom}}$ also relies on the results of [FF3, F], in which the existence of (0.1) was proved, and so it does not give us an alternative proof of the isomorphism (0.1). However, it helps us understand better the geometric meaning of this isomorphism and the role of Wakimoto modules in it. This is the first main result of this paper.

0.2. Next, we analyze the Wakimoto modules from the point of view of the equivalences of categories that appeared in our approach to the local geometric Langlands correspondence in [FG2, FG4].

Let us denote by $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}$ the category of discrete $\widehat{\mathfrak{g}}$-modules at the critical level on which the center $\mathfrak{Z}_{\mathfrak{g}}$ acts through its quotient $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} = \mathrm{Fun}(\mathrm{Op}_{\check{G}}^{\mathrm{reg}})$, where $\mathrm{Op}_{\check{G}}^{\mathrm{reg}} \subset \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ is the space of $\check{\mathfrak{g}}$-opers on the formal disc $\mathcal{D}$. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}^{I^0}$ be its subcategory of I^0-equivariant $\widehat{\mathfrak{g}}$-modules, where I^0 is the radical of the Iwahori subgroup $I \subset G((t))$. The algebra $\mathrm{Fun}(\mathrm{Op}_{\check{G}}^{\mathrm{reg}})$ acts on the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}^{I^0}$ in a natural way, so we may think of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}^{I^0}$ as "fibered" over the space $\mathrm{Op}_{\check{G}}^{\mathrm{reg}}$. In this Introduction, in order to simplify our notation, we will restrict ourselves to a "fiber" of this category, denoted by $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\chi}^{I^0}$, over a particular $\check{\mathfrak{g}}$-oper χ. The objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\chi}^{I^0}$ are I^0-equivariant $\widehat{\mathfrak{g}}$-modules at the critical level on which the center $\mathfrak{Z}_{\mathfrak{g}}$ acts through the character determined by χ. (In the main body of the paper we will work over the base $\mathrm{Op}_{\check{G}}^{\mathrm{reg}}$.)

We have constructed in [FG4] an equivalence of categories

$$(0.2) \qquad\qquad \Gamma^{\mathrm{Hecke}} : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} –\mathrm{mod}^{I^0} \overset{\sim}{\longrightarrow} \widehat{\mathfrak{g}}_{\mathrm{crit}} –\mathrm{mod}_{\chi}^{I^0}$$

(this equivalence is canonically defined for each trivialization of the flat $\check{G}$-bundle underlying the oper χ, which we will assume fixed in what follows). Here $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} –\mathrm{mod}^{I^0}$ is the category of I^0-equivariant critically twisted right D-modules on the affine Grassmannian Gr_G which satisfy the Hecke eigensheaf property (see [FG4], Sect. 1.1, and Sect. 3.2 below for the precise definition).

In addition, there is another equivalence of categories

$$(0.3) \qquad\qquad D\big(\mathrm{QCoh}((\mathrm{Fl}^{\check{G}})^{DG})\big) \simeq D\big(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} –\mathrm{mod}\big)^{I^0}.$$

Here $\mathrm{Fl}^{\check{G}}$ is the flag variety of $\check{G}$, and $D\big(\mathrm{QCoh}((\mathrm{Fl}^{\check{G}})^{DG})\big)$ is the derived category of complexes of quasi-coherent sheaves over the DG-scheme

$$(\mathrm{Fl}^{\check{G}})^{DG} := \mathrm{Spec}\Big(\mathrm{Sym}_{\mathcal{O}_{\mathrm{Fl}^{\check{G}}}}(\Omega^1(\mathrm{Fl}^{\check{G}})[1])\Big).$$

This DG-scheme can be realized as the derived Cartesian product $\widetilde{\check{\mathfrak{g}}} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt}$, where $\mathrm{pt} \to \check{\mathfrak{g}}$ corresponds to the point $0 \in \check{\mathfrak{g}}$, and $\widetilde{\check{\mathfrak{g}}} = \{(x, \check{\mathfrak{b}}) | x \in \check{\mathfrak{b}} \subset \check{\mathfrak{g}}\}$ is Grothendieck's alteration. The equivalence (0.3) follows from the results of [ABG], albeit in a somewhat indirect way.

Combining (0.2) and (0.3) we obtain an equivalence

$$(0.4) \qquad\qquad D\big(\mathrm{QCoh}((\mathrm{Fl}^{\check{G}})^{DG})\big) \simeq D(\widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_\chi^{I^0}).$$

The existence of such an equivalence is a corollary of the Main Conjecture 6.1.1 of [FG2] (see the Introduction to [FG4] for more details).

We have a natural direct image functor

$$(0.5) \qquad\qquad D\big(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\big) \to D\big(\mathrm{QCoh}((\mathrm{Fl}^{\check{G}})^{DG})\big),$$

and the second main objective of this paper is to describe explicitly its composition with (0.4), which is a functor

$$(0.6) \qquad\qquad \mathsf{G} : D\big(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\big) \to D(\widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_\chi^{I^0}).$$

(Note that unlike (0.4), the functor G is not an equivalence.)

The functor G of (0.6) turns out to be closely related to Wakimoto modules, as we shall presently explain. This relationship confirms the basic property of the equivalence (0.4) (and the more general equivalence of the Main Conjecture 6.1.1 of [FG2]), conjectured in [FG2], Sect. 6.1.

0.3. Recall that $\mathrm{Fl}^{\check{G}}$ has a stratification by Schubert cells $\mathrm{Fl}^{\check{G}}_w$, where w runs over the Weyl group W of $\check{G}$. In [FG2], Sect. 3.6 we have explained that $\mathrm{Fl}^{\check{G}}_w$ is the parameter space for Wakimoto modules at the critical level (having a fixed central character χ), and of "highest weight" $w(\rho) - \rho$.

More precisely, for every w we have a functor $_w\mathbb{W}$ from the category $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})_w$ of quasi-coherent sheaves on $\mathrm{Fl}^{\check{G}}$, which are *set-theoretically* supported on $\mathrm{Fl}^{\check{G}}_w$, to the category $\widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_\chi^{I^0}$. By definition, $_w\mathbb{W}$ is a kind of semi-infinite induction functor that is embodied by the Wakimoto module construction.

The second main result of this paper, Theorem 4.8, asserts that, up to a certain twist, the above functor $_w\mathbb{W}$ is isomorphic to the restriction of the functor G of (0.6) to $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})_w \subset \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$. In other words, the functor G "glues" the functors $_w\mathbb{W}$, which are defined for each $w \in W$ separately, into a single functor.

This fact has a number of interesting representation-theoretic implications. For example, let $\mathcal{T}_1, \mathcal{T}_2$ be two quasi-coherent sheaves on $\mathrm{Fl}^{\check{G}}$, supported set-theoretically on two different Schubert cells $\mathrm{Fl}^{\check{G}}_{w_1}$ and $\mathrm{Fl}^{\check{G}}_{w_2}$. Suppose that $\mathcal{T}_1 \to \mathcal{T}_2$ is a morphism between them.

Since G is a functor, we obtain a homomorphism $_{w_1}\mathbb{W}(\mathcal{T}_1) \to {}_{w_2}\mathbb{W}(\mathcal{T}_2)$ of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules. The existence of such a homomorphism is not obvious from the point of view of Wakimoto modules. However, examples of such homomorphisms have already existed:

If we take $\mathcal{T}_1$ to be the structure sheaf on the big cell $\mathrm{Fl}^{\check{G}}_1 \subset \mathrm{Fl}^{\check{G}}$, and $\mathcal{T}_2$ to be the quasi-coherent sheaf, underlying the D-module of distributions on a Schubert cell of codimension

1, the resulting morphism between the corresponding Wakimoto modules is the "screening operator" of [FF3].

As an application of the above results, we use the Cousin-Grothendieck resolution of the structure sheaf of $\mathrm{Fl}^{\check{G}}$ to construct a resolution of the vacuum module $\mathbb{V}_{\mathrm{crit}}$ at the critical level in terms of the Wakimoto modules, corresponding to distributions along the Schubert cells. The existence of such a resolution was conjectured in [FF3].

0.4.   Let us now explain the structure of the part of the paper that analyzes Wakimoto modules and the functor G.

In Sect. 3 we study the functor

$$\mathsf{E}: D\big(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\big) \to D\big(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}\big)^{I^0},$$

obtained by composing the functor (0.5) and the equivalence (0.3). The composition of E and the functor Γ^{Hecke} of (0.2) is the functor G of Sect. 0.6.

The main idea is that the functor E is fixed essentially uniquely by the condition that it respects the action of the group $\check{G}$ on both categories, where on the LHS this action comes from the $\check{G}$-action on $\mathrm{Fl}^{\check{G}}$, and on the RHS this action is as in [FG4], Sect. 2.1.

In Sect. 4 we collect some basic facts about Wakimoto modules, as well as some material from [FG4], and state our main result, Theorem 4.8.

Sections 5, 6 and 7 are devoted to the proof of Theorem 4.8. In Sect. 5 we treat a particular case of Theorem 4.8, namely the one of $w = 1$ and the quasi-coherent sheaf $\mathcal{O}_{\mathrm{Fl}_1^{\check{G}}}$, which corresponds to the vacuum Wakimoto module $\mathbb{W}_{\mathrm{crit},0}$. From this particular case, in Sect. 6 we derive the statement of Theorem 4.8 for the quasi-coherent sheaf $\mathcal{O}_{\mathrm{Fl}_w^{\check{G}}}$ for any w.

In Sect. 7 we derive the general case of Theorem 4.8 up to a certain twisting functor that acts from the category $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})_w$ to itself. To determine this twist, we need to analyze the action of the *renormalized enveloping algebra* $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ introduced in [BD].

In Sect. 8 we review the definition of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ and make a digression describing the behavior of certain natural Lie algebroids on the schemes $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ and $\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ under the isomorphism $\mathrm{map}_{\mathrm{geom}}$.

In Sect. 9 we analyze the interaction of the above algebroids and the Wakimoto functor, which will allow us to finish the proof of Theorem 4.8.

Finally, in Sect. 10 we give another construction of some Wakimoto modules at the critical level by a procedure of renormalization. In particular, the Wakimoto modules appearing in the Cousin-Grothendieck resolution mentioned above can be obtained in this way.

1. COMPARISON OF TWO MORPHISMS

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and G the connected simply-connected algebraic group with the Lie algebra $\mathfrak{g}$. We shall fix a Borel subgroup $B \subset G$. Denote by $\check{G}$ the Langlands dual group of G, and let $\check{\mathfrak{g}}$ be its Lie algebra. The group $\check{G}$ comes equipped with a Borel subgroup $\check{B} \subset \check{G}$.

Now let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ be the affine Kac-Moody algebra associated to the critical inner product κ_{crit} and $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod the category of discrete $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules (see [FG2]). Its objects are $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules in which every vector is annihilated by the Lie subalgebra $\mathfrak{g} \otimes t^n\mathbb{C}[[t]]$ for sufficiently large n. Let

$$\mathbb{V}_{\mathrm{crit}} := \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{C})$$

denote the vacuum module, which is an object of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod. Denote by $\mathfrak{Z}_{\mathfrak{g}}$ the topological commutative algebra that is the center of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod. Let $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ denote its "regular" quotient, i.e., the quotient modulo the annihilator of $\mathbb{V}_{\mathrm{crit}}$. It is known (see [FF3, F]) that

$$(1.1) \qquad \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} = \mathrm{End}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}_{\mathrm{crit}}) = (\mathbb{V}_{\mathrm{crit}})^{G[[t]]}.$$

Our goal in this section is to reprove a fundamental result of Beilinson and Drinfeld [BD], which compares two morphisms from $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ to the scheme $\mathrm{Op}_{\widehat{\mathfrak{g}}}^{\mathrm{reg}}$ of opers on the formal disc $\mathrm{Spec}(\mathbb{C}[[t]])$: one is the Feigin-Frenkel isomorphism [FF3, F], and the other is a morphism constructed by geometric means in [BD].

1.1. **Recollections.** Let X be a smooth complex curve. For a point $x \in X$ we denote by $\hat{\mathcal{O}}_x$ (resp., $\hat{\mathcal{K}}_x$) the corresponding completed local ring (resp., field), and write $\mathcal{D}_x = \mathrm{Spec}(\hat{\mathcal{O}}_x)$, $\mathcal{D}_x^{\times} = \mathrm{Spec}(\hat{\mathcal{K}}_x)$.

To $\mathfrak{g}$ and an invariant inner product κ on $\mathfrak{g}$ one associates a chiral algebra $\mathcal{A}_{\mathfrak{g},\kappa,X}$ on X (see [CHA]). Denote the chiral algebra corresponding to the critical inner product κ_{crit} by $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$. Consider the category of chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-modules, supported at a given point $x \in X$.

Let us choose a formal coordinate t at x. This choice defines a (tautological) equivalence between the above category and $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod, preserving the forgetful functor to the category of vector spaces. In particular, the vacuum object, i.e., the fiber $\mathcal{A}_{\mathfrak{g},\mathrm{crit},x}$ of $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$ at x, corresponds under this equivalence to $\mathbb{V}_{\mathrm{crit}}$. For this reason, when we view $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$ as a chiral module over itself we will sometimes denote it by $\mathbb{V}_{\mathrm{crit},X}$.

1.2. **A digression.** Set $\mathcal{D} = \mathrm{Spec}(\mathbb{C}[[t]])$ and $\mathcal{D}^{\times} = \mathrm{Spec}(\mathbb{C}((t)))$. Let $\mathrm{Aut}(\mathcal{D})$ be the group scheme of automorphisms of $\mathbb{C}[[t]]$ that preserve the maximal ideal. Let $\mathrm{Der}(\mathcal{D})$ be the Lie algebra of all derivations of $\mathbb{C}[[t]]$; it has a basis formed by elements $L_i = -t^{i+1}\partial_t$, $i \geq -1$. Note that $\mathrm{Lie}(\mathrm{Aut}(\mathcal{D}))$ is a subalgebra of codimension 1 in $\mathrm{Der}(\mathcal{D})$; the quotient is spanned by the image of L_{-1}.

We recall a general construction assigning to $\mathrm{Aut}(\mathcal{D})$-modules (resp., $(\mathrm{Der}(\mathcal{D}), \mathrm{Aut}(\mathcal{D}))$-modules) quasi-coherent $\mathcal{O}$-modules (resp., D-modules) on a smooth curve X (see, e.g., [FB] for more details).

Let Coord_X denote the $\mathrm{Aut}(\mathcal{D})$-torsor over X, whose fiber over $x \in X$, denoted by Coord_x, is the scheme of continuous isomorphisms between $\hat{\mathcal{O}}_x$ and $\mathbb{C}[[t]]$, preserving the maximal ideal (equivalently, "formal coordinates" at x). Note that the tautological $\mathrm{Aut}(\mathcal{D})$-action on Coord_X extends to a $\mathrm{Der}(\mathcal{D})$-action.

Let V be a representation of $\mathrm{Aut}(\mathcal{D})$. We form the associated $\mathcal{O}$-module on X, denoted V_X, by setting

$$\mathsf{V}_X := \mathrm{Coord}_X \overset{\mathrm{Aut}(\mathcal{D})}{\times} \mathsf{V}.$$

We will call such $\mathcal{O}$-modules *natural*. In other words, the fiber of V_X at any x is identified with V for every choice of a formal coordinate t near x. The $\mathcal{O}$-module V_X carries a natural action of the Lie algebra of vector fields on X by Lie derivatives, that we we will denote by $\xi, \mathsf{v} \mapsto \mathrm{Lie}_\xi(\mathsf{v})$.

For example, for an integer n, consider the character of $\mathrm{Aut}(\mathcal{D})$ given by $\mathrm{Aut}(\mathcal{D}) \twoheadrightarrow \mathbb{G}_m \overset{n}{\to} \mathbb{G}_m$, where the first arrow corresponds to the action of $\mathrm{Aut}(\mathcal{D})$ on the cotangent space to $\mathcal{D}$ at the origin. The corresponding natural $\mathcal{O}$-module identifies with ω_X^n, where ω_X is the canonical line bundle on X and the superscript "n" denotes the nth tensor power.

If V is a Harish-Chandra $(\mathrm{Der}(\mathcal{D}), \mathrm{Aut}(\mathcal{D}))$-module, then V_X acquires a natural left D-module structure. Let v_X be a local section of V_X near x, and let us choose a formal coordinate at x,

thereby identifying V_x with V; let $\mathsf{v} \in \mathsf{V}$ be the value of v_X at x, and let ξ be a vector field defined in a neighborhood of x. We have the following relation:

$$(1.2) \qquad\qquad \left(\mathrm{Lie}_\xi(\mathsf{v}_X) - \xi \cdot \mathsf{v}_X \right)_x = \xi(\mathsf{v}),$$

where $\xi \cdot \mathsf{v}_X$ refers to the left D-module structure on V_X, and $\xi(\mathsf{v})$ to the $\mathrm{Der}(D)$-action on V.

The chiral algebra $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$ itself is natural in the above sense (see [FB], Ch. 19), and most chiral modules over it that we will consider in this paper will also be natural. Such modules are the same as $\mathrm{Aut}(\mathcal{D})$-equivariant $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules. Thus, statements concerning such modules V_X (for all X simultaneously) are equivalent to $\mathrm{Aut}(\mathcal{D})$-equivariant statements concerning V.

1.3. Let $\mathfrak{z}_{\mathfrak{g},X}$ be the center of $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$, regarded as a commutative D-algebra. A choice of a coordinate as above identifies the fiber $\mathfrak{z}_{\mathfrak{g},x}$ at x with $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$. The topological commutative algebra $\widehat{\mathfrak{z}}_{\mathfrak{g},x}$, corresponding to $\mathfrak{z}_{\mathfrak{g},X}$ and $x \in X$ (see [CHA] Sect. 3.6.18), identifies with $\mathfrak{Z}_{\mathfrak{g}}$.

Let $\mathrm{Op}_{\check{\mathfrak{g}},X}$ be the D-scheme of $\check{\mathfrak{g}}$-opers on X introduced in [BD] (in this paper we follow the notation of [FG2]). Its fiber over $x \in X$ is the scheme $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}_x)$ of $\check{\mathfrak{g}}$-opers on the disc $\mathcal{D}_x$ at x. Again, a choice of a coordinate identifies $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}_x)$ with $\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$, the scheme of $\check{\mathfrak{g}}$-opers on $\mathcal{D}$, as subschemes in $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}_x^\times) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)$.

According to the results of [FF3, F], there exists a canonical isomorphism of D-schemes over X:

$$(1.3) \qquad\qquad \mathrm{map}_{\mathrm{alg}} : \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \overset{\sim}{\longrightarrow} \mathrm{Op}_{\check{\mathfrak{g}},X}\,.$$

In particular, we have isomorphisms

$$\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},x}) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}_x), \qquad \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}\,.$$

In Sect. 1.6 we will recall an important property of the isomorphism (1.3) that fixes it uniquely.

On the other hand, in [BD] a different map

$$\mathrm{map}_{\mathrm{geom}} : \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \to \mathrm{Op}_{\check{\mathfrak{g}},X}$$

was constructed, using the affine Grassmannian and the geometric Satake equivalence. We recall its definition below. We will then prove the following result:

Theorem 1.4. *The morphisms* $\mathrm{map}_{\mathrm{alg}}$ *and* $\mathrm{map}_{\mathrm{geom}}$ *coincide.*

This theorem was proved in [BD] by showing that these maps are compatible with the action of the Lie algebroids $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$ and $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$ (see [FG2], Sect. 4), defined on their left and right hand sides, respectively, and that a map satisfying this property is essentially unique.

Here we give a different proof in which we use the Wakimoto modules at the critical level and a key property of $\mathrm{map}_{\mathrm{alg}}$ established in [FF3, F] which we mentioned above.

1.5. **Definition of** $\mathrm{map}_{\mathrm{geom}}$. Let us recall the definition of the map $\mathrm{map}_{\mathrm{geom}}$ from [BD]. Let $\mathrm{Gr}_G = G((t))/G[[t]]$ be the affine Grassmannian. This is a strict ind-scheme; in particular it makes sense to consider the category $\mathrm{D}(\mathrm{Gr}_G)\mathrm{-mod}$ of right D-modules on Gr_G.

We have a natural action of $G[[t]]$ on Gr_G (given by left multiplication), such that the orbit of every finite-dimensional subscheme of $G[[t]]$ is still finite-dimensional. This insures that the category of $G[[t]]$-equivariant right D-modules on Gr_G is well-defined; we will denote this category by $\mathrm{D}(\mathrm{Gr}_G)\mathrm{-mod}^{G[[t]]}$.

The category $\mathrm{D}(\mathrm{Gr}_G)\mathrm{-mod}^{G[[t]]}$ is known to be semi-simple. Its irreducible objects can be described as follows. For $\check{\lambda} \in \check{\Lambda}^+$ (here, and in the rest of this paper, $\check{\Lambda}^+$ denotes the set of dominant coweights of G), let $\mathrm{Gr}_G^{\check{\lambda}} \subset \mathrm{Gr}_G$ be the $G[[t]]$-orbit of the point $\check{\lambda}(t) \in G((t))$, and

let $\overline{\mathrm{Gr}}_G^{\check{\lambda}}$ be its closure. The irreducible objects of $\mathrm{D}(\mathrm{Gr}_G)\,\text{–mod}^{G[[t]]}$ are the irreducible right D-modules $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^{\check{\lambda}}}$ corresponding to the strata $\overline{\mathrm{Gr}}_G^{\check{\lambda}}$.

By the geometric Satake equivalence (see [MV]), the category $\mathrm{D}(\mathrm{Gr}_G)\,\text{–mod}^{G[[t]]}$ has a natural structure of tensor category under convolution, and as such it is equivalent to $\mathrm{Rep}(\check{G})$. We will denote by $V \mapsto \mathcal{F}_V$ the corresponding tensor functor $\mathrm{Rep}(\check{G}) \to \mathrm{D}(\mathrm{Gr}_G)\,\text{–mod}^{G[[t]]}$. For $V = V^{\check{\lambda}}$, the irreducible $\check{G}$-representation of highest weight $\check{\lambda}$, the corresponding object $\mathcal{F}_{V^{\check{\lambda}}}$ is by definition isomorphic to $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^{\check{\lambda}}}$. For example, if $V = \mathbb{C}$ is the trivial representation, the corresponding D-module $\mathcal{F}_V$ is the δ-function D-module δ_{1,Gr_G} at the unit point of Gr_G.

Recall now that over Gr_G there exists a canonical line bundle $\mathcal{L}_{\mathrm{crit}}$; it is $G[[t]]$-equivariant. The action of $\mathfrak{g}((t))$ on Gr_G lifts to an action of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ on local sections of $\mathcal{L}_{\mathrm{crit}}$ (with the central element of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ mapping to the identity).

Thus, we can consider the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod}$ of $\mathcal{L}_{\mathrm{crit}}$-twisted right D-modules on Gr_G, and its $G[[t]]$-equivariant counterpart, denoted $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod}^{G[[t]]}$. The functor $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}_{\mathrm{crit}}$ defines an equivalence between the two categories (but this equivalence, of course, does not commute with the functor of global sections). By a slight abuse of notation, for $V \in \mathrm{Rep}(\check{G})$ we will denote by $\mathcal{F}_V$ also the corresponding irreducible object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod}^{G[[t]]}$.

We have the global sections functor

$$\Gamma : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\,\text{–mod}\,.$$

The main result of [FG1] asserts that this functor is exact and faithful.

In [BD] it is shown (using the results of [FF3, F]) that for $V \in \mathrm{Rep}(\check{G})$ there exists a locally free $(\mathrm{Der}(\mathcal{D}), \mathrm{Aut}(\mathcal{D}))$-equivariant $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}$-module $\mathcal{V}_{\mathfrak{z}}$, such that

$$(1.4) \qquad \Gamma(\mathrm{Gr}_G, \mathcal{F}_V) \simeq \mathcal{V}_{\mathfrak{z}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathbb{V}_{\mathrm{crit}}.$$

Moreover, the assignment $V \mapsto \mathcal{V}_{\mathfrak{z}}$ extends to a tensor functor from $\mathrm{Rep}(\check{G})$ to the category of locally free sheaves $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}$. By the Tannakian formalism, it defines a $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{z}}$ on $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}})$, equivariant with respect to the pair $(\mathrm{Der}(\mathcal{D}), \mathrm{Aut}(\mathcal{D}))$, such that

$$\mathcal{V}_{\mathfrak{z}} = \mathcal{P}_{\check{G},\mathfrak{z}} \overset{\check{G}}{\times} V.$$

We will now consider the relative versions of the above objects over the curve X. Let $\mathrm{Gr}_{G,X}$ be the global version of the affine Grassmannian: this is an ind-scheme over X whose fiber at $x \in X$ is $\mathrm{Gr}_{G,x} = G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x)$. Globally, we have:

$$\mathrm{Gr}_{G,X} \simeq \mathrm{Coord}_X \overset{\mathrm{Aut}(\mathcal{D})}{\times} \mathrm{Gr}_G,$$

where Coord_X is as in Sect. 1.2. In addition, $\mathrm{Gr}_{G,X}$ is endowed with a connection along X. By the same construction, the line bundle $\mathcal{L}_{\mathrm{crit}}$ gives rise to a line bundle $\mathcal{L}_{\mathrm{crit},X}$ on $\mathrm{Gr}_{G,X}$, and the connection on $\mathrm{Gr}_{G,X}$ lifts to $\mathcal{L}_{\mathrm{crit},X}$.

Let $\mathrm{Jets}(G)_X$ denote the group D-scheme of jets of sections of the group scheme $X \times G$ over X; the fiber $\mathrm{Jets}(G)_x$ of $\mathrm{Jets}(G)_X$ at $x \in X$ is by definition $G(\hat{\mathcal{O}}_x)$.

We will consider the category $\mathrm{D}(\mathrm{Gr}_{G,X})_{\mathrm{crit}}\,\text{–mod}$ (resp., $\mathrm{D}(\mathrm{Gr}_{G,X})_{\mathrm{crit}}\,\text{–mod}^{\mathrm{Jets}(G)_X}$) of critically twisted (resp., $\mathrm{Jets}(G)_X$-equivariant) right D-modules on $\mathrm{Gr}_{G,X}$. Since each object of the form $\mathcal{F}_V \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod}^{G[[t]]}$ for $V \in \mathrm{Rep}(\check{G})$ is $\mathrm{Aut}(\mathcal{D})$-equivariant, it gives rise to a well-defined object of $\mathrm{D}(\mathrm{Gr}_{G,X})_{\mathrm{crit}}\,\text{–mod}^{\mathrm{Jets}(G)_X}$, which we will denote by $\mathcal{F}_{V,X}$.

For any object $\mathcal{F}_X \in \mathrm{D}(\mathrm{Gr}_{G,X})_{\mathrm{crit}}$ –mod, the direct image of $\mathcal{F}_X$, considered as a quasi-coherent sheaf, onto X, is naturally a right D-module. Moreover, it has a natural structure of chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module. By a slight abuse of notation we denote it by $\Gamma(\mathrm{Gr}_{G,X}, \mathcal{F}_X)$.

From (1.4) we obtain that, globally over X, there exist $\mathfrak{z}_{\mathfrak{g},X}$-modules $\mathcal{V}_{\mathfrak{z},X}$, endowed with a connection along X, such that

$$(1.5) \qquad\qquad \Gamma(\mathrm{Gr}_{G,X}, \mathcal{F}_{V,X}) \simeq \mathcal{V}_{\mathfrak{z},X} \underset{\mathfrak{z}_{\mathfrak{g},X}}{\otimes} \mathbb{V}_{\mathrm{crit},X},$$

and a $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{z},X}$ on $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X})$ with a connection along X such that

$$\mathcal{V}_{\mathfrak{z},X} = \mathcal{P}_{\check{G},\mathfrak{z},X} \overset{\check{G}}{\times} V.$$

Furthermore, this $\check{G}$-torsor is endowed with a reduction to the Borel subgroup $\check{B} \subset \check{G}$, as we shall presently explain.

Consider the object $\mathcal{F}_{V^{\check{\lambda}},X} \in \mathrm{D}(\mathrm{Gr}_{G,X})_{\mathrm{crit}}$ –mod$^{\mathrm{Jets}(G)_X}$, corresponding to a dominant coweight $\check{\lambda}$ of G. By the semi-simplicity of the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$ –mod$^{G[[t]]}$, this D-module equals the 0-th cohomology of the *-extension of the *constant* critically twisted right D-module on the corresponding $\mathrm{Jets}(G)_X$-orbit $\mathrm{Gr}_{G,X}^{\check{\lambda}} \subset \mathrm{Gr}_{G,X}$, the latter being by definition the line bundle $\mathcal{L}_{\mathrm{crit},X}|_{\mathrm{Gr}_{G,X}^{\check{\lambda}}} \otimes \Omega^{\mathrm{top}}_{\mathrm{Gr}_{G,X}^{\check{\lambda}}}$.

Recall now, that according to [BD], there exists a canonical isomorphism

$$(1.6) \qquad\qquad \mathcal{L}_{\mathrm{crit},X}|_{\mathrm{Gr}_{G,X}^{\check{\lambda}}} \otimes \Omega^{\mathrm{top}}_{\mathrm{Gr}_{G,X}^{\check{\lambda}}} \simeq p^*(\omega_X^{\langle \rho, \check{\lambda} \rangle})|_{\mathrm{Gr}_{G,X}^{\check{\lambda}}},$$

where $p : \mathrm{Gr}_{G,X} \to X$ is the canonical projection, and ω_X is the line bundle of 1-forms on X. Hence we obtain a map $\omega_X^{\langle \rho, \check{\lambda} \rangle} \to \Gamma(\mathrm{Gr}_{G,X}, \mathcal{F}_{V,X})^{\mathrm{Jets}(G)_X}$. Using the isomorphism (1.5) and the fact that

$$(\mathbb{V}_{\mathrm{crit},X})^{\mathrm{Jets}(G)_X} = \mathfrak{z}_{\mathfrak{g},X},$$

which follows from (1.1), we obtain a map

$$(1.7) \qquad\qquad \kappa^{\check{\lambda}} : \mathfrak{z}_{\mathfrak{g},X} \underset{\mathcal{O}_X}{\otimes} \omega_X^{\langle \rho, \check{\lambda} \rangle} \to \mathcal{V}_{\mathfrak{z},X}^{\check{\lambda}}.$$

Note that $\mathfrak{z}_{\mathfrak{g},X} \underset{\mathcal{O}_X}{\otimes} \omega_X^{\langle \rho, \check{\lambda} \rangle}$ is a plain line bundle over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X})$, i.e., it has no connection along X.

It is easy to see that the system of maps $\kappa^{\check{\lambda}}$ satisfies the Plücker relations (for the definition, see, e.g., [FG4], Sect. 4.1) and therefore defines a reduction of $\mathcal{P}_{\check{G},\mathfrak{z},X}$ to $\check{B}$. Moreover, it is shown in [BD] that this reduction to $\check{B}$ satisfies the oper condition relative to the connection along X on $\mathcal{P}_{\check{G},\mathfrak{z},X}$.

This defines the desired morphism $\mathrm{map}_{\mathrm{geom}} : \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X}) \to \mathrm{Op}_{\check{\mathfrak{g}},X}$.

1.6. **The defining property of** $\mathrm{map}_{\mathrm{alg}}$. Let us now recall from [FF3, F] a property of the morphism $\mathrm{map}_{\mathrm{alg}}$ that fixes it uniquely (we will follow the notation of [FG2]). Let $\mathfrak{H}_{\mathrm{crit},X}$ be the commutative D-algebra on X, introduced in [FG2], Sect. 10.6. By definition, $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$ is the D-scheme of induction parameters for Wakimoto modules. (We will review what this means below.)

According to the results of [FF3, F] (see also [FG2], Sect. 10.8), we have a canonical map of D-algebras

$$\varphi : \mathfrak{z}_{\mathfrak{g},X} \to \mathfrak{H}_{\mathrm{crit},X}.$$

Let $\check{H} = \check{B}/[\check{B}, \check{B}]$ be the Cartan group of $\check{G}$. Let ω_X^ρ be the $\check{H}$-torsor over X, induced from the $\mathbb{G}_m$-torsor, corresponding to the line bundle ω_X via the co-character $\rho : \mathbb{G}_m \to \check{H}$.

Consider the D-scheme $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$, classifying connections on the $\check{H}$-torsor ω_X^ρ.

Both $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$ and $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$ are naturally torsors with respect to the D-scheme classifying $\mathfrak{h}^* = \check{\mathfrak{h}}$-valued one-forms on X. According to [CHA], 2.8.17, we have a canonical isomorphism

$$(1.8) \qquad \mathrm{map}_{\mathrm{alg}}^M : \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \simeq \mathrm{Conn}_{\check{H}}(\omega_X^\rho),$$

respecting the torsor structure.

Finally, recall from [F], Sect. 10.3 (see also [FG2], Sect. 3.3) that we have a canonical map $\mathrm{MT} : \mathrm{Conn}_{\check{H}}(\omega_X^\rho) \to \mathrm{Op}_{\check{\mathfrak{g}},X}$, called the Miura transformation.

The defining property of $\mathrm{map}_{\mathrm{alg}}$ proved in [FF3, F] is that the diagram

$$(1.9)$$
$$
\begin{array}{ccc}
\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) & \xrightarrow{\ \tau \circ \mathrm{map}_{\mathrm{alg}}^M\ } & \mathrm{Conn}_{\check{H}}(\omega_X^\rho) \\
\varphi \downarrow & & \mathrm{MT} \downarrow \\
\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}},X) & \xrightarrow{\ \mathrm{map}_{\mathrm{alg}}\ } & \mathrm{Op}_{\check{\mathfrak{g}},X}
\end{array}
$$

is commutative, where τ denotes the automorphism of $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$, induced by the automorphism of $\check{H}$, given by $\check{\lambda} \mapsto -w_0(\check{\lambda})$. (Note that this automorphism is well-defined on $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$, since $-w_0(\rho) = \rho$.) Moreover, this property determines the isomorphism $\mathrm{map}_{\mathrm{alg}}$ uniquely.

Our strategy of the proof of Theorem 1.4 will be as follows: we will construct another map

$$(1.10) \qquad \mathrm{map}_{\mathrm{geom}}^M : \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \simeq \mathrm{Conn}_{\check{H}}(\omega_X^\rho),$$

in the spirit of the above construction of $\mathrm{map}_{\mathrm{geom}}$, for which the diagram

$$(1.11)$$
$$
\begin{array}{ccc}
\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}^M\ } & \mathrm{Conn}_{\check{H}}(\omega_X^\rho) \\
\varphi \downarrow & & \mathrm{MT} \downarrow \\
\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}},X) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}\ } & \mathrm{Op}_{\check{\mathfrak{g}},X}
\end{array}
$$

is manifestly commutative. In addition, we will see that the maps $\mathrm{map}_{\mathrm{geom}}^M$ and $\tau \circ \mathrm{map}_{\mathrm{alg}}^M$ coincide, thereby implying Theorem 1.4.

2. A geometric construction of Miura opers

The construction of the map $\mathrm{map}_{\mathrm{geom}}^M$ utilizes the Wakimoto modules introduced in [W, FF1, FF2, F]. In this paper we will mostly follow the notation of [FG2], Part III.

2.1. Let $\widehat{\mathfrak{H}}_{\mathrm{crit},x}$ be the topological commutative algebra, corresponding to the commutative chiral algebra $\mathfrak{H}_{\mathrm{crit},X}$ and the point $x \in X$. There exists a canonically defined topological commutative algebra $\widehat{\mathfrak{H}}_{\mathrm{crit}}$, acted on by $\mathrm{Aut}(\mathcal{D})$, such that every choice of a coordinate t at x defines an isomorphism $\widehat{\mathfrak{H}}_{\mathrm{crit},x} \simeq \widehat{\mathfrak{H}}_{\mathrm{crit}}$.

In fact, $\widehat{\mathfrak{H}}_{\mathrm{crit}}$ is the completed universal enveloping algebra of a certain canonical central (and in fact commutative) extension $\widehat{\mathfrak{h}}_{\mathrm{crit}}$ of $\mathfrak{h}((t))$. In what follows for $\mu \in \mathfrak{h}^*$, we will denote by $\pi_{\mathrm{crit},\mu}$ the $\widehat{\mathfrak{H}}_{\mathrm{crit}}$-module

$$\mathrm{Ind}_{\mathfrak{h}[[t]]}^{\widehat{\mathfrak{h}}_{\mathrm{crit}}}(\mathbb{C}^\mu),$$

where $\mathbb{C}^\mu$ is the 1-dimensional representation of $\mathfrak{h}[[t]]$, corresponding to the character $\mathfrak{h}[[t]] \twoheadrightarrow \mathfrak{h} \xrightarrow{\mu} \mathbb{C}$.

Since $\widehat{\mathfrak{H}}_{\mathrm{crit}}$ is commutative, $\pi_{\mathrm{crit},\mu}$ is in fact a quotient algebra, which we will also denote by $\mathfrak{H}_{\mathrm{crit},\mu}^{\mathrm{RS},\mu}$. For $\mu = 0$, we have $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},0} = \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$, which identifies with the fiber $\mathfrak{H}_{\mathrm{crit},x}$ at x. The corresponding global chiral $\mathfrak{H}_{\mathrm{crit},X}$-module, denoted $\pi_{\mathrm{crit},X}$, is, by definition, the vacuum module $\mathfrak{H}_{\mathrm{crit},X}$.

2.2. Wakimoto modules.
In [FG2], Sect. 11.3 we defined a functor $\mathbb{W}^{w_0}$ from the category of chiral $\mathfrak{H}_{\mathrm{crit},X}$-modules to that of chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-modules. In the present paper we will only consider Wakimoto modules "of type w_0", so we will omit the superscript w_0 from the notation.

In particular, we obtain a functor

$$\mathbb{W} : \widehat{\mathfrak{H}}_{\mathrm{crit}} \text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}} \text{–mod}.$$

Set $\mathbb{W}_{\mathrm{crit},\mu} := \mathbb{W}(\pi_{\mathrm{crit},w_0(\mu)})$. We remark that in [F] this module was denoted by W_{μ,κ_c}, and in [FG2] by $\mathbb{W}_{\mathrm{crit},\mu}^{w_0}$. For $\mu = 0$ we will denote by $\mathbb{W}_{\mathrm{crit},X}$ the chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module $\mathbb{W}(\pi_{\mathrm{crit},X}^0)$.

Assume now that μ is integral. Then by [FG2], Sect. 11, the Wakimoto module $\mathbb{W}_{\mathrm{crit},\mu}$ is integrable with respect to the Iwahori subgroup $I \subset G[[t]]$, i.e., it is naturally an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}} \text{–mod}^I$, where the latter denotes the category of I-integrable representations of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$.

2.3.   The starting point of our construction of the map $\mathrm{map}_{\mathrm{geom}}^M$ is the following observation.

Let $\mathrm{Fl}_G^{\mathrm{aff}} = G((t))/I$ be the affine flag scheme of G. Let $\mathrm{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}} \text{–mod}$ be the category of critically twisted right D-modules on $\mathrm{Fl}_G^{\mathrm{aff}}$ and $\mathrm{D}(\mathrm{Fl}_G^{\mathrm{aff}})_\kappa \text{–mod}^I$ the corresponding I-equivariant category. Recall that to $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}} \text{–mod}^I$ and $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}} \text{–mod}^I$ we can associate their convolution

$$\mathcal{F} \star \mathcal{M} \in \mathrm{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}} \text{–mod})^I$$

(see [FG2], Sect. 22.5 for precise definitions). In particular, for each integral coweight $\check{\lambda}$ of G we have an object $j_{\check{\lambda},*}$ of $\mathrm{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}} \text{–mod}^I$ (see [FG2], Sect. 12.1). It is the *-extension of the twisted right D-module on $\mathrm{Fl}_G^{\mathrm{aff}}$ corresponding to the constant sheaf on the I–orbit $I \cdot \check{\lambda}(t) \subset \mathrm{Fl}_G^{\mathrm{aff}}$. The following proposition is a generalization of [FG2], Corollary 13.4.2:

Proposition 2.4. *For $\check{\lambda} \in \check{\Lambda}^+$, there exists a canonical isomorphism*

$$j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},\mu} \simeq \mathbb{W}_{\mathrm{crit},\mu} \otimes \mathfrak{l}_{\mu+\rho}^{\check{\lambda}},$$

respecting the action of $\mathrm{Aut}(\mathcal{D})$, where $\mathfrak{l}_{\mu+\rho}^{\check{\lambda}}$ is a 1-dimensional $\mathrm{Aut}(\mathcal{D})$-module, corresponding to the character

$$\mathrm{Aut}(\mathcal{D}) \twoheadrightarrow \mathbb{G}_m \xrightarrow{\langle \mu+\rho, \check{\lambda} \rangle} \mathbb{G}_m.$$

Proof. By [FG2], Corollary 13.4.1, there exists a *non-canonical* isomorphism

$$j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},\mu} \simeq \mathbb{W}_{\mathrm{crit},\mu}.$$

Hence, by Proposition 13.1.2 of *loc. cit.*, there exists an $\mathrm{Aut}(\mathcal{D})$-equivariant line bundle $\mathcal{L}_{\mathfrak{H},\mu}^{\check{\lambda}}$ over $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\check{\mu})})$ and an isomorphism

$$j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},\mu} \simeq \mathbb{W}_{\mathrm{crit},\mu} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\mu)}}{\otimes} \mathcal{L}_{\mathfrak{H},\mu}^{\check{\lambda}}.$$

Consider the embedding $\mathbb{G}_m \hookrightarrow \mathrm{Aut}(\mathcal{D})$ given by loop rotations $c \in \mathbb{G}_m \mapsto (t \mapsto c \cdot t)$. Since the weights of the resulting action of $\mathbb{G}$ on $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\mu)}$ are non-positive, and $\mathbb{C} \subset \mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\mu)}$ is

the weight zero subspace, the line bundle $\mathcal{L}_{\mathfrak{H},\mu}^{\check{\lambda}}$ is of the form $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\mu)} \otimes \mathfrak{l}_{\mu+\rho}^{\check{\lambda}}$, where $\mathfrak{l}_{\mu+\rho}^{\check{\lambda}}$ is a 1-dimensional representation of $\mathrm{Aut}(\mathcal{D})$.

The group $\mathrm{Aut}(\mathcal{D})$ acts on $\mathfrak{l}_{\mu+\rho}^{\check{\lambda}}$ via its projection onto $\mathbb{G}_m$. We have to show that the corresponding character of $\mathbb{G}_m$ is given by the integer $\langle \mu + \rho, \check{\lambda} \rangle$. I.e., we have to compare the natural actions of $L_0 = -t\partial_t \in \mathrm{Der}(\mathcal{D})$ on $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},\mu}$ and $\mathbb{W}_{\mathrm{crit},\mu}$, and show that their difference equals the above integer.

Let $\kappa_\hbar$ be a one-parameter deformation of κ_{crit} away from the critical level, and consider the Wakimoto modules $\mathbb{W}_{\hbar,\mu_\hbar} := \mathbb{W}_{\kappa_\hbar,\mu_\hbar}$, where $\mu_\hbar$ is an $\hbar$-family of weights such that $\mu_\hbar \bmod \hbar = \mu$. Each of these modules is endowed with an action of L_0 via the Segal-Sugawara construction. When we shift this action by $\frac{1}{2} \cdot (\kappa_\hbar - \kappa_{\mathrm{crit}})^{-1}(\mu_\hbar, \mu_\hbar + 2 \cdot \rho)$, this limiting action at $\kappa_\hbar = \kappa_{\mathrm{crit}}$ equals the natural one on $\mathbb{W}_{\mathrm{crit},\mu}$.

By [FG2], Proposition 12.5.1,

$$\widetilde{j}_{\kappa_\hbar,\check{\lambda}} \underset{I^0}{\star} \mathbb{W}_{\hbar,\mu_\hbar} \simeq \mathbb{W}_{\hbar,\mu_\hbar - \nu_\hbar},$$

where $\nu_\hbar$ is the weight such that $\langle \nu_\hbar, \check{\mu} \rangle = (\kappa_\hbar - \kappa_{\mathrm{crit}})(\check{\lambda}, \check{\mu})$. Here I^0 is the pro-unipotent radical of I and we use the convolution functor, denoted by $\underset{I^0}{\star}$, between I^0-equivariant twisted D-modules on $\widetilde{\mathrm{Fl}}_G = G((t))/I^0$ and I^0-equivariant $\widehat{\mathfrak{g}}_{\kappa_\hbar}$-modules (note that since the weights on $\mathbb{W}_{\hbar,\mu_\hbar}$ are non-integral for generic $\hbar$, it is not I-equivariant, but only I^0-equivariant). The corresponding I^0-equivariant D-module $\widetilde{j}_{\kappa_\hbar,\check{\lambda}}$ on $\widetilde{\mathrm{Fl}}_G$ was defined in [FG2], Sect. 12.1, where the connection between the convolution functors $\widetilde{j}_{\kappa_\hbar,\check{\lambda}} \underset{I^0}{\star} M$ and $j_{\check{\lambda},*} \star M$ is also discussed.

We obtain that in the limit $\kappa_\hbar \to \kappa_{\mathrm{crit}}$ the resulting L_0 action on $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},\mu}$ differs from that on $\mathbb{W}_{\mathrm{crit},\mu}$ by the limit of

$$\frac{1}{2} \cdot (\kappa_\hbar - \kappa_{\mathrm{crit}})^{-1}(\mu_\hbar - \nu_\hbar, \mu_\hbar - \nu_\hbar + 2 \cdot \rho) - \frac{1}{2} \cdot (\kappa_\hbar - \kappa_{\mathrm{crit}})^{-1}(\mu_\hbar, \mu_\hbar + 2 \cdot \rho),$$

which equals $-\langle \mu + \rho, \check{\lambda} \rangle$.

$\square$

2.5. **Definition of** $\mathrm{map}'^{M}_{\mathrm{geom}}$. Let us consider the chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module $\mathbb{W}_{\mathrm{crit},X}$; it is obtained from $\mathbb{W}_{\mathrm{crit},0}$ by the procedure described in Sect. 1.2.

In a similar fashion, we can consider a global version of $\mathrm{Fl}_G^{\mathrm{aff}}$, and of the twisted D-modules $j_{\check{\lambda},*}$; we will denote the latter by $j_{\check{\lambda},*,X}$.

Consider now the chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module $j_{\check{\lambda},*,X} \star \mathbb{W}_{\mathrm{crit},X}$. From Proposition 13.1.2 of [FG2], we obtain that there exists a line bundle $\mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}}$ over $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$, endowed with a connection along X, and an isomorphism:

$$(2.1) \qquad j_{\check{\lambda},*,X} \star \mathbb{W}_{\mathrm{crit},X} \simeq \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathbb{W}_{\mathrm{crit},X}.$$

The data $\check{\lambda} \mapsto \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}}$ give rise a $\check{H}$-torsor on $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$, once we define isomorphisms

$$(2.2) \qquad \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathcal{L}_{\mathfrak{H},X}^{\check{\mu}} \simeq \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}+\check{\mu}}$$

compatible with the triple tensor products. These are defined by the requirement that the diagram

$$
\begin{array}{ccc}
j_{\check{\mu},*,X} \star (j_{\check{\lambda},*,X} \star \mathbb{W}_{\mathrm{crit},X}) & \longrightarrow & j_{\check{\mu},*,X} \star (\mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathbb{W}_{\mathrm{crit},X}) \\
\downarrow & & \sim\downarrow \\
j_{\check{\lambda}+\check{\mu},*,X} \star \mathbb{W}_{\mathrm{crit},X} & & \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} (j_{\check{\mu},*,X} \star \mathbb{W}_{\mathrm{crit},X}) \\
\downarrow & & \downarrow \\
\mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}+\check{\mu}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathbb{W}_{\mathrm{crit},X} & \longrightarrow & \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathcal{L}_{\mathfrak{H},X}^{\check{\mu}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathbb{W}_{\mathrm{crit},X}
\end{array}
$$

be commutative.

From Proposition 2.4, we obtain that as a plain $\mathfrak{H}_{\mathrm{crit},X}$-module (i.e., disregarding the connection along X), $\mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}}$ is isomorphic to $\omega_X^{\langle\rho,\check{\lambda}\rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X} \otimes \mathfrak{l}^{\check{\lambda}}$, where $\mathfrak{l}^{\check{\lambda}}$ is a (constant) line.

Thus, we obtain that the line bundles $\omega_X^{\langle\rho,\check{\lambda}\rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X}$ on $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$ acquire connections along X. These connections are compatible with the isomorphisms (2.2). These data give rise to a morphism of D-schemes

$$
\mathrm{map'}_{\mathrm{geom}}^{M} : \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \to \mathrm{Conn}_{\check{H}}(\omega_X^{\rho}).
$$

We will establish the following:

Proposition 2.6. *The map* $\mathrm{map'}_{\mathrm{geom}}^{M}$ *coincides with the map* $\tau \circ \mathrm{map}_{\mathrm{alg}}^{M}$.

A proof of this proposition will be given at the end of this section. We proceed with the proof of Theorem 1.4.

2.7. **Definition of the map** $\mathrm{map}_{\mathrm{geom}}^{M}$. Let $\mathrm{MOp}_{\check{G},X}$ be the (non-affine) D-scheme of Miura opers, as defined in [F] (see also [FG2], Sect. 3). By definition, it classifies opers on X, endowed with an additional data of reduction of the corresponding $\check{G}$-bundle to the subgroup $\check{B}^- \subset \check{G}$, compatible with the connection. (Here $\check{B}^- \subset \check{G}$ is an opposite Borel subgroup that we choose once and for all.)

Let $\mathrm{MOp}_{\check{G},\mathrm{gen},X} \subset \mathrm{MOp}_{\check{G},X}$ be the open subscheme, classifying generic Miura opers, as defined in [F], Sect. 10.3 (see also [FG2], Sect. 3.3). The genericity condition is that the above reduction to $\check{B}^-$ is at all points of X in the generic position with respect to the reduction to $\check{B}$, given by the oper structure.

By *loc. cit.* there exists a canonical isomorphism of D-schemes

$$
\tag{2.3} \mathrm{MOp}_{\check{G},\mathrm{gen},X} \simeq \mathrm{Conn}_{\check{H}}(\omega_X^{\rho}).
$$

Thus, constructing a morphism $\mathrm{map}_{\mathrm{geom}}^{M} : \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \to \mathrm{Conn}_{\check{H}}(\omega_X^{\rho})$ is equivalent to constructing a morphism

$$
\tag{2.4} \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \to \mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X}.
$$

By definition, to define a map as in Sect. 2.4, we need to construct a $\check{G}$-bundle $\mathcal{P}_{\check{G},\mathfrak{H},X}$ on $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$, equipped with a connection along X, endowed with an oper structure, and a reduction to $\check{B}^-$, which is compatible with connection, and which is in generic relative position with respect to the reduction to $\check{B}$, given by the oper structure.

We define the $\check{G}$-bundle $\mathcal{P}_{\check{G},\mathfrak{H},X}$ on $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$ with its oper structure as the pull-back from $\mathcal{P}_{\check{G},\mathfrak{Z},X}$ on $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g},X})$ via the map φ. To define a reduction to $\check{B}^{-}$ on $\mathcal{P}_{\check{G},\mathfrak{H},X}$ we proceed as follows:

We have the canonical embedding of chiral $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules

$$\phi : \mathbb{V}_{\mathrm{crit}} \to \mathbb{W}_{\mathrm{crit},0}.$$

For $\check{\lambda} \in \check{\Lambda}^{+}$ consider the I-orbit $\mathrm{Gr}_{G}^{\check{\lambda},I} = I \cdot \check{\lambda}(t)$ in Gr_{G}. Let $j_{\check{\lambda},\mathrm{Gr}_{G},*}$ be the *-extension of the right D-module on this orbit corresponding to the constant sheaf. Observe that $\mathrm{Gr}_{G}^{\check{\lambda},I}$ is open and dense in the $G[[t]]$-orbit $\mathrm{Gr}_{G}^{\check{\lambda}} = G[[t]] \cdot \check{\lambda}(t)$, so that we have a map $\mathcal{F}_{V^{\check{\lambda}}} \to j_{\check{\lambda},\mathrm{Gr}_{G},*}$. Furthermore, under the projection $\mathrm{Fl}_{G}^{\mathrm{aff}} \to \mathrm{Gr}_{G}$ the I-orbit $I \cdot \check{\lambda}(t) \subset \mathrm{Fl}_{G}^{\mathrm{aff}}$ is mapped to $\mathrm{Gr}_{G}^{\check{\lambda},I}$ one-to-one. Hence, we have an isomorphism

$$j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \simeq j_{\check{\lambda},\mathrm{Gr}_{G},*} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}},$$

where $\underset{G[[t]]}{\star}$ denotes the convolution functor between the category of twisted D-modules on Gr_{G} and the category of $G[[t]]$-equivariant $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules (to which $\mathbb{V}_{\mathrm{crit}}$ belongs). Consider the composition:

$$(2.5) \qquad \Gamma(\mathrm{Gr}_{G}, \mathcal{F}_{V}) \simeq \mathcal{F}_{V^{\check{\lambda}}} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}} \to j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \xrightarrow{j_{\check{\lambda},*} \star \phi} j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit}}.$$

Hence, using (1.4) and Proposition 2.4, we obtain a map

$$(2.6) \qquad \mathcal{V}_{\mathfrak{Z}}^{\check{\lambda}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathbb{V}_{\mathrm{crit}} \to \mathcal{L}_{\mathfrak{H}}^{\check{\lambda}} \otimes \mathbb{W}_{\mathrm{crit},0},$$

where $\mathcal{L}_{\mathfrak{H}}^{\check{\lambda}} := \mathcal{L}_{\mathfrak{H},0}^{\check{\lambda}} \simeq \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \otimes \mathfrak{l}_{\rho}^{\check{\lambda}}$ is an in the proof of Proposition 2.4

However, by [FG2], Proposition 10.7.1, the map

$$\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to \mathbb{W}_{\mathrm{crit},0}$$

induces an isomorphism of the spaces of $G[[t]]$-invariants

$$\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to (\mathbb{W}_{\mathrm{crit},0})^{G[[t]]}.$$

Therefore, the map (2.6) gives rise to a map of $\mathfrak{H}_{\mathrm{crit},X}$-modules, compatible with the connection:

$$(2.7) \qquad \kappa_{\mathfrak{H}}^{-,\check{\lambda}} : \mathcal{V}_{\mathfrak{Z}}^{\check{\lambda}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to \mathcal{L}_{\mathfrak{H}}^{\check{\lambda}}.$$

The same construction can be performed over the base X, and we obtain a map

$$(2.8) \qquad \mathcal{V}_{\mathfrak{Z},X}^{\check{\lambda}} \underset{\mathfrak{z}_{\mathfrak{g},X}}{\otimes} \mathbb{V}_{\mathrm{crit},X} \to \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X}}{\otimes} \mathbb{W}_{\mathrm{crit},X},$$

and hence a map

$$(2.9) \qquad \kappa_{\mathfrak{H}}^{-,\check{\lambda}} : \mathcal{V}_{\mathfrak{Z},X}^{\check{\lambda}} \underset{\mathfrak{z}_{\mathfrak{g},X}}{\otimes} \mathfrak{H}_{\mathrm{crit},X} \to \mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}}.$$

The maps $\kappa_{\check{\mathfrak{H}}}^{-,\check{\lambda}}$ for $\check{\lambda} \in \check{\Lambda}^+$ satisfy the Plücker equations since the diagrams

$$
\begin{array}{ccccc}
\mathcal{F}_{V^{\check{\lambda}}} \star \mathcal{F}_{V^{\check{\mu}}} & \longrightarrow & j_{\check{\lambda},\mathrm{Gr}_G,*} \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check{\mu}}} & \overset{\sim}{\longrightarrow} & j_{\check{\lambda},*} \star \mathcal{F}_{V^{\check{\mu}}} \\
\downarrow & & & & \downarrow \\
\mathcal{F}_{V^{\check{\lambda}+\check{\mu}}} & \longrightarrow & j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*} & \overset{\sim}{\longrightarrow} & j_{\check{\lambda},*} \star j_{\check{\mu},\mathrm{Gr}_G,*}
\end{array}
$$

are clearly commutative. Therefore the system of maps $\{\kappa_{\check{\mathfrak{H}}}^{-,\check{\lambda}}\}$ defines a reduction of $\mathcal{P}_{\check{G},\mathfrak{H},X}$ to $\check{B}^-$. By construction, this reduction is horizontal with respect to the connection.

In order to show that this $\check{B}^-$-reduction is in generic relative position with the oper $\check{B}$-reduction it suffices to prove the following:

Proposition 2.8. *The composed arrow*

$$
\omega_X^{\langle \rho, \check{\lambda} \rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X} \overset{\kappa^{\check{\lambda}}}{\rightarrow} \mathcal{V}_{\mathfrak{Z},X}^{\check{\lambda}} \underset{\mathfrak{z}_{\mathfrak{g},X}}{\otimes} \mathfrak{H}_{\mathrm{crit},X} \overset{\kappa_{\check{\mathfrak{H}}}^{-,\check{\lambda}}}{\rightarrow} \mathcal{L}_{\check{\mathfrak{H}},X}^{\check{\lambda}} \simeq \omega_X^{\langle \rho, \check{\lambda} \rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X} \otimes \mathfrak{l}^{\check{\lambda}}
$$

is an isomorphism, which is induced by a trivialization of the $\check{H}$-torsor $\{\check{\lambda} \mapsto \mathfrak{l}^{\check{\lambda}}\}$.

Let us assume this proposition and finish the proof of Theorem 1.4.

First, Proposition 2.8 implies that the data $(\mathcal{P}_{\check{G},\mathfrak{H},X}, \kappa^{\check{\lambda}}, \kappa_{\check{\mathfrak{H}}}^{-,\check{\lambda}})$ define a generic Miura oper over $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X})$. This gives rise to a map $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \to \mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X}$, which we compose with the identification (2.3) to produce the sought-after $\mathrm{map}_{\mathrm{geom}}^M$,

The diagram (1.11) is commutative, since the map $\mathrm{MT} : \mathrm{Conn}_{\check{H}}(\omega_X^\rho) \to \mathrm{Op}_{\check{\mathfrak{g}},X}$ is by definition the composition of (2.3) and the tautological projection $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X} \to \mathrm{Op}_{\check{\mathfrak{g}},X}$.

In view of Proposition 2.6, it remains to show that the map $\mathrm{map}_{\mathrm{geom}}^M$, constructed above coincides with $\mathrm{map}'^M_{\mathrm{geom}}$.

To see that, we recall from [F], Sect. 10.3 (see also [FG2], Sect. 3.3) that the map $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X} \overset{\sim}{\longrightarrow} \mathrm{Conn}_{\check{H}}(\omega_X^\rho)$ of (2.3) is defined as follows:

Given a Miura oper, the genericity assumption implies that the $\check{H}$-bundle, induced from the $\check{B}^-$-bundle, is isomorphic to ω_X^ρ, and hence, the latter acquires a connection.

Therefore, the connection along X on every line bundle $\omega_X^{\langle \rho, \check{\lambda} \rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X}$, corresponding to $\mathrm{map}_{\mathrm{geom}}^M$, equals the one arising from the composed isomorphism of Proposition 2.8. The latter equals, by definition, to the connection on $\omega_X^{\langle \rho, \check{\lambda} \rangle} \underset{\mathcal{O}_X}{\otimes} \mathfrak{H}_{\mathrm{crit},X}$, corresponding to the map $\mathrm{map}'^M_{\mathrm{geom}}$.

This completes the proof of Theorem 1.4.

2.9. **Proof of Proposition 2.8.** It is enough to show that for a fixed point $x \in X$ the composition

(2.10) $\qquad \omega_x^{\langle \rho, \check{\lambda} \rangle} \to \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}}) \to j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0} \simeq \omega_x^{\langle \rho, \check{\lambda} \rangle} \otimes \mathfrak{l}^{\check{\lambda}} \otimes \mathbb{W}_{\mathrm{crit},0}$

is an isomorphism onto the sub-space $\omega_x^{\langle \rho, \check{\lambda} \rangle} \otimes \mathfrak{l}^{\check{\lambda}} \subset \mathbb{W}_{\mathrm{crit},0}$, corresponding to the generating vector of $\mathbb{W}_{\mathrm{crit},0}$.

In fact, we claim that it is enough to show that the composition in (2.10) is non-zero. Indeed, according to [BD], Proposition 8.1.5, the image of $\omega_x^{\langle \rho, \check{\lambda} \rangle}$ in $\Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}})$ equals the subspace on which the operator L_0 acts with the eigenvalue $-\langle \rho, \check{\lambda} \rangle$ (all other eigenvalues of L_0 being strictly greater). Similarly, the generating vector of $\mathbb{W}_{\mathrm{crit},0}$ spans the subspace corresponding to the zero eigenvalue of L_0.

To show the non-vanishing of (2.10) we proceed as follows. Let us apply to the two sides of the morphism

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}}) \to j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$$

the semi-infinite cohomology functor $H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0\big)$ (see [FG2], Sect. 18).

As in [FG2], Sect. 18.3, we have a commutative diagram

$$
\begin{array}{ccc}
H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}}) \otimes \Psi_0\big) & \longrightarrow & H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], (j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}) \otimes \Psi_0\big) \\
\sim \downarrow & & \sim \downarrow \\
H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathrm{crit}} \otimes \Psi_{\check{\lambda}}\big) & \xrightarrow{\ \phi\ } & H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}_{\mathrm{crit},0} \otimes \Psi_{\check{\lambda}}\big).
\end{array}
$$

It is easy to see that under the left vertical map the image of

$$\omega_x^{\langle \rho, \check{\lambda}\rangle} \to \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}})^{G[[t]]} \to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}}) \otimes \Psi_0\big)$$

is mapped to the one-dimensional vector space spanned by the image of the canonical generator of $\mathbb{V}_{\mathrm{crit}}$ in

$$\mathbb{V}_{\mathrm{crit}}^{G[[t]]} \to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathrm{crit}} \otimes \Psi_{\check{\lambda}}\big).$$

Under the bottom horizontal map the latter goes to the image of the the canonical generator of $\mathbb{W}_{\mathrm{crit},0}$ in

$$\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}_{\mathrm{crit},0} \otimes \Psi_{\check{\lambda}}\big),$$

and, in particular, it is non-zero.

2.10. **Proof of Proposition 2.6.** We claim that it is enough to show that the map

$$\tau \circ \mathrm{map'}\,^M_{\mathrm{geom}} : \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},X}) \to \mathrm{Conn}_{\check{H}}(\omega_X^{\rho})$$

respects the torsor structure with respect to the D-scheme of $\mathfrak{h}^*$-valued one-forms on X.

Indeed, the morphism $\mathrm{map}^M_{\mathrm{alg}}$ has this property and is an isomorphism, by definition. Therefore the difference of the two maps $\tau \circ \mathrm{map'}\,^M_{\mathrm{geom}} - \mathrm{map}^M_{\mathrm{alg}}$ can be regarded as an $\mathfrak{h}^*$-valued one-form, canonically attached to the curve X. In particular, this one-form would be invariant under Lie derivatives by all vector fields, which implies that it is equal to zero.

To verify the required property of the map $\mathrm{map'}\,^M_{\mathrm{geom}}$ with respect to the torsor structure we need to check the following:

Consider the action of $\mathrm{Der}(\mathcal{D})$ on $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0} \simeq \mathfrak{l}_{\rho}^{\check{\lambda}} \otimes \mathbb{W}_{\mathrm{crit},0}$ at some point $x \in X$, given by the formula (1.2). This action preserves the subspace of $\mathfrak{g}[[t]]$-invariant vectors.

Recall that $L_{-1} \in \mathrm{Der}(\mathcal{D})$ denotes the element $-\partial_t$. Let $\mathbf{w}$ be an element from the 1-dimensional vector space $\mathfrak{l}_{\rho}^{\check{\lambda}} \subset j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$.

We need to show that

$$L_{-1} \cdot \mathbf{w} \bmod \mathbb{C} \cdot \mathbf{w} = (w_0(\check{\lambda}) \otimes t^{-1}) \cdot \mathbf{w},$$

where we identify $(\mathfrak{h} \otimes t^{-1} \cdot \mathbb{C}[[t]])/\mathbb{C}[[t]]$ with a sub-space of $\widehat{\mathfrak{h}}_{\mathrm{crit}}/\mathfrak{h}[[t]] \subset \mathfrak{H}_{\mathfrak{h}}^{\mathrm{reg}}$. The latter is a straightforward calculation, performed below.

We may realize the action of $\mathrm{Der}(\mathcal{D})$ on $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$ as the limit $\kappa_{\hbar} \to \kappa_{\mathrm{crit}}$ of its actions on

$$\widetilde{j}_{\kappa_{\hbar},\check{\lambda},*} \underset{I^0}{\star} \mathbb{W}_{\hbar,0} \simeq \mathbb{W}_{\hbar,-\nu_{\hbar}},$$

given by the Segal-Sugawara construction, where $\nu_{\hbar}$ is as in the proof of Proposition 2.4.

Let us note also that in order to show that two elements of $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq \mathbb{W}_{\mathrm{crit},0}^{G[[t]]}$ are equal, it is sufficient to analyze their images in

$$\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}_{\mathrm{crit},0}\big).$$

Let us regard $H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)),\mathfrak{n}[[t]],\mathbb{W}_{\hbar,-\nu_\hbar}\big)$ as acted on by the chiral algebra $\mathfrak{H}'_{\kappa_\hbar}$ in the notation of [FG2], Sect. 10.2. Let $\widehat{\mathfrak{h}}'_\hbar$ be the corresponding central extension of $\mathfrak{h}((t))$.

We have a canonical isomorphism

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)),\mathfrak{n}[[t]],\mathbb{W}_{\hbar,-\nu_\hbar}\big) \simeq \pi'_{\hbar,-\nu_\hbar} := \mathrm{Ind}_{\mathfrak{h}[[t]]}^{\widehat{\mathfrak{h}}'_\hbar}(\mathbb{C}^{-\nu_\hbar}),$$

compatible with the $\mathrm{Der}(\mathcal{D})$-action.

Note that for $\kappa_\hbar = \kappa_{\mathrm{crit}}$ the action of $\mathfrak{H}^{\mathrm{reg}}_{\mathrm{crit}}$ on $H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)),\mathfrak{n}[[t]],\mathbb{W}_{\mathrm{crit},0}\big)$, coming from its action on $\mathbb{W}_{\mathrm{crit},0}$, coincides with the one given by the isomorphism of chiral algebras $\mathfrak{H}'_{\mathrm{crit},X} \simeq \mathfrak{H}_{\mathrm{crit},X}$ of [FG2], 10.6, composed with the automorphism, induced by τ.

To summarize, we need to compute the action of L_{-1}, given by the Segal-Sugawara construction, on the $\widehat{\mathfrak{h}}'_\hbar$-module $\pi'_{\hbar,-\nu_\hbar}$ in the limit $\kappa_\hbar \to \kappa_{\mathrm{crit}}$.

If $\mathbf{v}_\hbar$ denotes the canonical generator in $\pi_{\hbar,-\nu_\hbar}$, we need to show that

$$L_{-1} \cdot \mathbf{v}_\hbar \bmod \mathbb{C} \cdot \mathbf{v}_\hbar = -(\check\lambda \otimes t^{-1}) \cdot \mathbf{v}_\hbar \bmod \hbar.$$

However, this coincides with the formula for the action of the operator L_{-1} on the Heisenberg algebra $\widehat{\mathfrak{h}}'_\hbar$ (see [F], Sect. 5.5). This completes the proof of Proposition 2.6.

3. From $\mathcal{O}$-modules on $\mathrm{Fl}^{\check{G}}$ to D-modules on Gr_G

3.1.   Let $\mathrm{Fl}^{\check{G}}$ be the flag variety of the group $\check{G}$, thought of as the quotient $\check{G}/\check{B}^-$. Consider the category $\mathrm{QCoh}\big(\mathrm{Fl}^{\check{G}}\big)$ of quasi-coherent sheaves on $\mathrm{Fl}^{\check{G}}$, and the corresponding derived category $D\big(\mathrm{QCoh}\big(\mathrm{Fl}^{\check{G}}\big)\big)$.

Consider the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$–mod of *Hecke eigensheaves* on Gr_G, introduced in [FG4], Sect. 2.1 (the definition will be recalled below).

In this section we will study the functor

$$\mathsf{E}: D^+\big(\mathrm{QCoh}\big(\mathrm{Fl}^{\check{G}}\big)\big) \to D^+\big(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}\big),$$

obtained by composing the equivalence (0.3) of [ABG] and the direct image functor (0.5), as was explained in the Introduction. The functor E will be left-exact.

We should remark, however, that in this paper we will not formally rely on the results of [ABG]. We will construct *a* functor E "from scratch", with *loc. cit.* serving as a guide.

We remark that the contents of this section have a significant intersection with Sect. 3.2.13 of [ABBGM] and Sect. 3 of [FG4].

3.2.   Let us first recall the definition of the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$–mod.[1]

Its objects are the data of $(\mathcal{F}, \alpha_V, \ \forall\, V \in \mathrm{Rep}(\check{G}))$, where $\mathcal{F}$ is an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod, and $\alpha_V, V \in \mathrm{Rep}(\check{G}))$, are isomorphisms of D-modules

$$\alpha_V : \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_V \xrightarrow{\ \sim\ } \underline{V} \otimes \mathcal{F},$$

(here $\underline{V}$ denotes the vector space underlying the representation V) such that the following two conditions are satisfied:

- If V is the trivial representations $\mathbb{C}$, then the morphism α_V is the identity map.

---

[1]We remark that in [FG2] we used the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$–mod when the group G was of the adjoint type. It is easy to see, however, that the two categories are equivalent (see [AG]).

- For $V, W \in \mathrm{Rep}(\check{G})$ and $U := V \otimes W$, the diagram

$$
\begin{array}{ccc}
\left(\mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_V\right) \underset{G[[t]]}{\star} \mathcal{F}_W & \xrightarrow{\;\sim\;} & \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_U \\[2mm]
\alpha_V \underset{G[[t]]}{\star} \mathrm{id}_{\mathcal{F}_W} \Big\downarrow & & \alpha_U \Big\downarrow \\[2mm]
\left(\underline{V} \otimes \mathcal{F}\right) \underset{G[[t]]}{\star} \mathcal{F}_W & & \underline{U} \otimes \mathcal{F} \\[2mm]
\sim \Big\downarrow & & \sim \Big\downarrow \\[2mm]
\underline{V} \otimes \left(\mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_W\right) & \xrightarrow{\;\mathrm{id}_{\underline{V}} \otimes \alpha_W\;} & \underline{V} \otimes \underline{W} \otimes \mathcal{F}
\end{array}
$$

is commutative.

Morphisms in this category between $(\mathcal{F}, \alpha_V)$ and $(\mathcal{F}', \alpha_V')$ are maps of D-modules $\phi : \mathcal{F} \to \mathcal{F}'$, and such that

$$(\mathrm{id}_{\underline{V}} \otimes \phi) \circ \alpha_V = \alpha_V' \circ (\phi \star \mathrm{id}_{\mathcal{F}_V}).$$

Note that this category carries a canonical action of the group $\check{G}$. Indeed, for $\mathbf{g} \in \check{G}$ and an object $(\mathcal{F}, \alpha_V) \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$ –mod we define a new object as $(\mathcal{F}, \mathbf{g}|_{\underline{V}} \cdot \alpha_V)$, where $\mathbf{g}|_{\underline{V}}$ denotes the automorphism by means of which $\mathbf{g}$ acts on the vector space underlying V.

3.3. Consider now the following general situation. Let $\mathcal{C}$ be an abelian category equipped with an action of $\check{G}$; let $\mathrm{act}^* : \mathcal{C} \to \mathrm{QCoh}(\check{G}) \otimes \mathcal{C}$ be the corresponding action functor (we refer the reader to [FG2], Sect. 20.1 where the corresponding notions are discussed in detail).

We will describe a general framework, in which one constructs a functor

$$\mathsf{F} : D^+(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})) \to D^+(\mathcal{C}),$$

compatible with the $\check{G}$-actions.

Suppose we are given a collection of $\check{G}$-equivariant objects $\mathcal{M}_{\check{\lambda}} \in \mathcal{C}$, $\check{\lambda} \in \check{\Lambda}^+$, and a collection of $\check{G}$-equivariant morphisms

$$\beta^{\check{\mu}, \check{\lambda}} : V^{\check{\mu}} \otimes \mathcal{M}_{\check{\lambda}} \to \mathcal{M}_{\check{\lambda} + \check{\mu}},$$

defined for $\check{\mu} \in \check{\Lambda}^+$, where the LHS is endowed with a $\check{G}$-equivariant structure via the diagonal action. We will assume that for $\check{\mu}_1, \check{\mu}_2 \in \check{\lambda} \in \check{\Lambda}^+$ the diagram

(3.1)
$$
\begin{array}{ccc}
V^{\check{\mu}_1} \otimes V^{\check{\mu}_2} \otimes \mathcal{M}_{\check{\lambda}} & \xrightarrow{\;\mathrm{id}_{V^{\check{\mu}_1}} \otimes \beta^{\check{\mu}_2, \check{\lambda}}\;} & V^{\check{\mu}_1} \otimes \mathcal{M}_{\check{\lambda} + \check{\mu}_2} \\[2mm]
\Big\downarrow & & \beta^{\check{\mu}+1, \check{\lambda}+\check{\mu}_2} \Big\downarrow \\[2mm]
V^{\check{\mu}_1 + \check{\mu}_2} \otimes \mathcal{M}_{\check{\lambda}} & \xrightarrow{\;\beta^{\check{\mu}_1 + \check{\mu}_2, \check{\lambda}}\;} & \mathcal{M}_{\check{\lambda} + \check{\mu}_1 + \check{\mu}_2}
\end{array}
$$

is commutative.

One example of the above situation arises when $\mathcal{C} = \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ and $\mathcal{M}_{\check{\lambda}}$ is taken to be $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}$, i.e., the $\check{G}$-equivariant line bundle on $\mathrm{Fl}^{\check{G}}$, attached to the weight $\check{\lambda}$. Our normalization is such that $\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}) = V^{\check{\lambda}}$.

We shall now perform the following procedure in this example that will allow us to expresses a large class of objects of $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ in terms of the line bundles $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}$:

Let $\mathcal{T}$ be a quasi-coherent sheaf on $\mathrm{Fl}^{\check{G}}$ obtained as the direct image of a quasi-coherent sheaf on an affine locally closed subscheme of $\mathrm{Fl}^{\check{G}}$.

For $\check{\mu} \in \check{\Lambda}^+$ consider the direct sums

$$\mathcal{T}^+ := \bigoplus_{\check{\lambda} \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}}) \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}$$

and

$$\mathcal{T}^{++} := \bigoplus_{\check{\mu}, \check{\lambda} \in \check{\Lambda}^+} \underline{V}^{\check{\mu}} \otimes \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}) \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}.$$

Note that we have a canonical map

(3.2) $$\mathcal{T}^+ \to \mathcal{T}.$$

In addition, we have two maps

(3.3) $$\mathcal{T}^{++} \rightrightarrows \mathcal{T}^+.$$

The first map comes from the canonical map

(3.4) $$\underline{V}^{\check{\mu}} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \to \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}+\check{\mu}},$$

tensored with the identity on each $\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}})$. The second map comes from the map

$$\underline{V}^{\check{\mu}} \otimes \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}) \to \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}}),$$

tensored with the identity map on $\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}$.

The proof of the following result is straightforward.

Lemma 3.4. *The map* (3.2) *identifies* $\mathcal{T}$ *with the co-equalizer of the map* (3.3).

We will now perform the same construction in a category $\mathcal{C}$ equipped with the above structures. For $\mathcal{T}$ as above consider two objects of $\mathcal{C}$ defined as:

$$\mathsf{F}(\mathcal{T})^+ := \bigoplus_{\check{\lambda} \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}}) \otimes \mathcal{M}_{\check{\lambda}}$$

and

$$\mathsf{F}(\mathcal{T})^{++} := \bigoplus_{\check{\mu}, \check{\lambda} \in \check{\Lambda}^+} \underline{V}^{\check{\mu}} \otimes \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}) \otimes \mathcal{M}_{\check{\lambda}}.$$

The maps $\beta^{\check{\mu}, \check{\lambda}}$ and (3.4) give rise to two morphisms:

(3.5) $$\mathsf{F}(\mathcal{T})^{++} \rightrightarrows \mathsf{F}(\mathcal{T})^+.$$

Define $\mathsf{F}(\mathcal{T})$ as the co-equalizer of the above maps. It is clear that if $\mathcal{T}$ is $\check{G}$ (or $\check{B}$)-equivariant, then so is $\mathsf{F}(\mathcal{T})$. Set

(3.6) $$\mathcal{M}_{w_0} := \mathsf{F}(\mathbb{C}_{w_0}),$$

where $\mathbb{C}_{w_0}$ is the sky-scraper at the $\check{B}$-invariant point $w_0 \in \mathrm{Fl}^{\check{G}}$. By the above, $\mathcal{M}_{w_0}$ is $\check{B}$-equivariant.

To extend the functor F to arbitrary quasi-coherent sheaves on $\mathrm{Fl}^{\check{G}}$, we need to make a digression and discuss the following general construction.

3.5. **Coherent convolutions.** Let $\mathcal{C}$ be as above, $\mathcal{M} \in \mathcal{C}$ an object equivariant with respect to $\check{B} \subset \check{G}$, and let $\mathcal{T}$ be a quasi-coherent sheaf on $\mathrm{Fl}^{\check{G}}$. Let us identify $\mathrm{Fl}^{\check{G}}$ with the quotient $\check{G}/\check{B}$, by the action of $\check{G}$ on the point $w_0 \in \mathrm{Fl}^{\check{G}} := \check{G}/\check{B}^-$ stabilized by $\check{B}$. Let $\widetilde{\mathcal{T}}$ be the $\check{B}$-equivariant quasi-coherent sheaf on $\check{G}$, corresponding to $\mathcal{T}$.

Consider the object $\widetilde{\mathcal{T}} \underset{\mathrm{Fun}(\check{G})}{\otimes} \mathrm{act}^*(\mathcal{M}) \in \mathrm{QCoh}(\check{G}) \otimes \mathcal{C}$. Regarded as an object of $\mathcal{C}$, it carries an action of $\check{B}$ by automorphisms, compatible with the action of the latter on $\mathrm{Fun}(\check{G})$ by right translations.

We define the convolution $\mathcal{T} * \mathcal{M} \in \mathcal{C}$ as

$$(3.7) \qquad \mathcal{T} * \mathcal{M} = \left(\widetilde{\mathcal{T}} \underset{\mathrm{Fun}(\check{G})}{\otimes} \mathrm{act}^*(\mathcal{M}) \right)^{\check{B}}.$$

Keeping $\mathcal{M} \in \mathcal{C}$ fixed, we can regard $\mathcal{T} \mapsto \mathcal{T} * \mathcal{M}$ as a functor $\mathrm{QCoh}(\check{G}) \to \mathcal{C}$. This functor is evidently left exact, and we will denote by $\mathcal{T} \overset{R}{*} \mathcal{M}$ its right derived functor.

It is easy to see that if $\mathcal{T}$ is isomorphic to the direct image of a quasi-coherent sheaf on an affine locally closed subscheme of $\mathrm{Fl}^{\check{G}}$, then we have an isomorphism $\mathcal{T} \overset{R}{*} \mathcal{M} \simeq \mathcal{T} * \mathcal{M}$.

Let us consider some examples of the above situation. Take first $\mathcal{C} = \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ with the natural action of $\check{G}$, and let $\mathcal{M}$ be $\mathbb{C}_{w_0}$–the skyscraper at the the point $w_0 \in \mathrm{Fl}^{\check{G}}$. Since this point is fixed by $\check{B}$, the object $\mathbb{C}_{w_0}$ is $\check{B}$-equivariant.

It is clear that in this case the functor $\mathcal{T} \mapsto \mathcal{T} * \mathbb{C}_{w_0}$ is tautologically isomorphic to the identity functor.

Suppose now that F is a left-exact functor $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}}) \to \mathcal{C}$, compatible with the actions of $\check{G}$ in the evident sense. Set $\mathcal{M}_{w_0} := \mathsf{F}(\mathbb{C}_{w_0})$. This object is $\check{B}$-equivariant, by transport of structure.

We then obtain a functorial isomorphism:

$$(3.8) \qquad \mathsf{F}(\mathcal{T}) \simeq \mathcal{T} * \mathcal{M}_{w_0}.$$

Finally, let us consider the case $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}$, and let us write down the convolution functors more explicitly.

Namely, for an object $\mathcal{M} := (\mathcal{F}, \alpha_V)$ to be $\check{B}$-equivariant means that $\mathcal{F}$, as an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, is endowed with an algebraic action of $\check{B}$ by automorphisms, and the morphisms

$$\alpha_V : \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_V \simeq \underline{V} \otimes \mathcal{F}$$

respect the $\check{B}$-action, where on the LHS the action comes by functoriality from the $\check{B}$-action on $\mathcal{F}$, and on the RHS it is diagonal with respect to the natural $\check{B}$-action on $\underline{V}$.

Then the object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, underlying $\mathcal{T} * \mathcal{M}$ is given by $\left(\widetilde{\mathcal{T}} \otimes \mathcal{F} \right)^{\check{B}}$. (It is also easy to write down the remaining data of $\mathcal{T} * \mathcal{M}$ making it into an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}$.)

3.6. Let us return to the framework of Sect. 3.3. Let $\mathcal{M}_{w_0}$ be given by (3.6). From the construction we have:

Lemma 3.7. *For $\mathcal{T}$, which is the direct image of a quasi-coherent sheaf on an affine locally closed subscheme of $\mathrm{Fl}^{\check{G}}$, there exists a canonical isomorphism*

$$\mathsf{F}(\mathcal{T}) \simeq \mathcal{T} \overset{R}{*} \mathcal{M}_{w_0}.$$

We now define the functor

$$\mathsf{F} : D^+\left(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\right) \to D^+(\mathcal{C})$$

by the formula

$$\mathsf{F}(\mathcal{T}) := \mathcal{T} \overset{R}{*} \mathcal{M}_{w_0}.$$

Lemma 3.7 implies that for $\mathcal{T}$ being the direct image of a quasi-coherent sheaf on an affine locally closed subscheme of $\mathrm{Fl}^{\check{G}}$, this definition agrees with the one of Sect. 3.3.

Let $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w_0}}$ denote the line equal to the fiber of the line bundle $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}$ at the point $w_0 \in \mathrm{Fl}^{\check{G}}$. We will regard it as equipped with an action of $\check{B}$ given by the character $w_0(\check{\lambda})$.

We have the following general assertion, valid in the set-up of Sect. 3.5, whose proof is straightforward:

Lemma 3.8. *Let $\mathcal{M}$ be a $\check{G}$-equivariant object of $\mathcal{C}$. Then for $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ and $\check{\mu} \in \check{\Lambda}$ there exists a canonical isomorphism*

$$\mathcal{T} \overset{R}{*} \left(\mathcal{L}^{\check{\mu}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}\right) \simeq R\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T} \otimes \mathcal{L}^{\check{\mu}}_{\mathrm{Fl}^{\check{G}}}) \otimes \mathcal{M}.$$

By construction, for $\check{\lambda} \in \check{\Lambda}^+$ we have a map

$$(3.9) \qquad \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\lambda}} \to \mathcal{M}_{w_0},$$

of $\check{B}$-equivariant objects of $\mathcal{C}$. Hence, by functoriality, and using Lemma 3.8, we obtain a map

$$(3.10) \qquad \mathcal{M}_{\check{\lambda}} \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{M}_{w_0} := \mathsf{F}(\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}).$$

We are going to show now that under some additional hypotheses the map (3.10) is in fact an isomorphism for large enough $\check{\lambda} \in \check{\Lambda}^+$. We will make the following two additional assumptions:

(A) The maps $\beta^{\check{\mu},\check{\lambda}} : V^{\check{\mu}} \otimes \mathcal{M}_{\check{\lambda}} \to \mathcal{M}_{\check{\lambda}+\check{\mu}}$ are surjective once $\check{\lambda}$ is deep enough in the dominant chamber. (In our main example, namely, $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$–mod, the surjectivity will hold for all $\check{\lambda}$ that are regular.)

(B) The objects $\mathcal{M}_{\check{\mu}}$ are Artinian, and for every $\check{B}$-equivariant object $\mathcal{M}$, which is Artinian as an object of $\mathcal{C}$, the higher cohomologies of $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{M}$ vanish for all $\check{\lambda}$ that are large enough (i.e., deep enough in the dominant chamber).

We start with the following

Lemma 3.9. *Assume that condition (A) holds. Then the map*

$$(3.11) \qquad \mathcal{L}^{-\check{\mu}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}} \to \mathcal{M}_{w_0}$$

is surjective for $\check{\mu}$ deep inside the dominant chamber.

Proof. By assumption (A), there exists $\check{\mu} \in \check{\Lambda}^+$ such that the maps $\beta^{\check{\nu},\check{\mu}}$ are surjective for all $\check{\nu} \in \check{\Lambda}^+$. Let $\check{\mu}' \in \check{\Lambda}^+$ be such that $\check{\mu}' - \check{\mu} =: \check{\nu} \in \check{\Lambda}^+$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}^{-\check{\mu}'}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}'} & \longrightarrow & \mathcal{M}_{w_0} \\
{\scriptstyle \beta^{\check{\nu},\check{\mu}}}\big\uparrow & & {\scriptstyle (3.11)}\big\uparrow \\
\mathcal{L}^{-\check{\mu}'}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \underline{V}^{\check{\nu}} \otimes \mathcal{M}_{\check{\mu}} & \longrightarrow & \mathcal{L}^{-\check{\mu}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}},
\end{array}
$$

and hence the image of the upper horizontal arrow is contained in that of (3.11), since the left vertical arrow is surjective.

For an arbitrary $\check{\mu}'' \in \check{\Lambda}^+$ we can find $\check{\nu}' \in \check{\Lambda}^+$ such that $\check{\mu}' := \check{\mu}'' + \check{\nu}'$ satisfies $\check{\mu}' - \check{\mu} \in \check{\Lambda}^+$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}^{-\check{\mu}'}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \underline{V}^{\check{\nu}'} \otimes \mathcal{M}_{\check{\mu}''} & \longrightarrow & \mathcal{L}^{-\check{\mu}''}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}''} \\
{\scriptstyle \beta^{\check{\nu}',\check{\mu}''}} \downarrow & & \downarrow \\
\mathcal{L}^{-\check{\mu}'}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}'} & \longrightarrow & \mathcal{M}_{w_0}.
\end{array}
$$

Since the upper horizontal arrow in this diagram (which comes from the map $\underline{V}^{\check{\nu}'} \otimes \mathcal{O}_{\mathrm{Fl}^{\check{G}}} \to \mathcal{L}^{\check{\nu}'}_{\mathrm{Fl}^{\check{G}}}$) is surjective, we obtain that the image of $\mathcal{L}^{-\check{\mu}''}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}''}$ in $\mathcal{M}_{w_0}$ is contained in that of $\mathcal{L}^{-\check{\mu}'}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}'}$, and the latter is contained in the image of (3.11), as we have seen above.

Since $\mathcal{M}_{w_0}$ is by definition the quotient of the direct sum $\underset{\check{\lambda} \in \check{\Lambda}^+}{\oplus} \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\lambda}}$, whose components map to $\mathcal{M}_{w_0}$ by means of (3.11), we obtain the statement of the lemma. $\square$

Proposition 3.10. *Assume that conditions (A) and (B) hold. Then the map (3.10) is an isomorphism for all $\check{\lambda} \in \check{\Lambda}^+$ that are large enough.*

Proof. By assumption (B) and Lemma 3.9, $\mathcal{M}_{w_0}$ is Artinian. Applying assumption (B) once more, we obtain that $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{M}_{w_0}$ has no higher cohomologies for $\check{\lambda}$ large enough.

Let us prove the surjectivity of the map

(3.12)
$$
\mathcal{M}_{\check{\lambda}} \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}_{w_0},
$$

provided that $\check{\lambda}$ is large.

Let $\mathcal{M}'$ denote the kernel of this map. Let $\check{\lambda}$ be large so that $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{M}'$ has no higher cohomologies. Assume also that $\check{\lambda} - \check{\mu}$ is dominant.

We obtain a short exact sequence

$$
0 \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}' \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * (\mathcal{L}^{-\check{\mu}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}}) \twoheadrightarrow \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}_{w_0} \to 0,
$$

that fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * (\mathcal{L}^{-\check{\mu}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}}) & \longrightarrow & \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}_{w_0} \\
{\scriptstyle \text{Lemma 3.8}} \downarrow & & \uparrow {\scriptstyle (3.12)} \\
\underline{V}^{\check{\lambda}-\check{\mu}} \otimes \mathcal{M}_{\check{\mu}} & \xrightarrow{\ \beta^{\check{\lambda}-\check{\mu},\check{\mu}}\ } & \mathcal{M}_{\check{\lambda}},
\end{array}
$$

implying surjectivity of the right vertical arrow for $\check{\lambda}$ as above.

Now we prove the injectivity of (3.12). Let $\mathcal{M}''$ denote the kernel of the map (3.9). By (B), we can find a finite collection of elements $(\check{\mu}_i, \check{\nu}_i)$, so that $\mathcal{M}''$ is contained in the image of $\underset{i}{\oplus} \underline{V}^{\check{\nu}_i} \otimes \mathcal{L}^{-\check{\mu}_i-\check{\nu}_i}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\mu}_i}$ under the difference of the two maps in (3.5). Let $\mathcal{M}'''$ denote the pre-image of $\mathcal{M}''$ in the above direct sum. Let us assume that $\check{\lambda}$ is such that $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{M}'''$ does not have higher cohomologies. To prove the injectivity of (3.12), it suffices to prove that the map

$$
\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}''' \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * \mathcal{M}'' \hookrightarrow \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} * (\mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w_0}} \otimes \mathcal{M}_{\check{\lambda}}) \simeq \mathcal{M}_{\check{\lambda}}
$$

is zero, but this follows from the commutativity of (3.1).

$\square$

3.11. **An explicit resolution.** Let us return to the case when $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ is the direct image of a quasi-coherent sheaf on an affine subvariety of $\check{G}$. For a non-negative integer i consider the following object of $\mathcal{C}$

$$\mathsf{F}(\mathcal{T})^{+(i)} := \bigoplus_{\check{\mu}_i,\dots,\check{\mu}_1,\check{\lambda}\in\check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}_1-\dots-\check{\mu}_i})\otimes\underline{V}^{\check{\mu}_1}\otimes\dots\otimes\underline{V}^{\check{\mu}_i}\otimes\mathcal{M}_{\check{\lambda}}.$$

We have $i+1$ maps $\mathfrak{d}_i^j : \mathsf{F}(\mathcal{T})^{+(i)} \to \mathsf{F}(\mathcal{T})^{+(i-1)}$, where $\mathfrak{d}_i^0$ comes from (3.4), $\mathfrak{d}_i^i$ comes from $\beta^{\check{\mu}_i,\check{\lambda}}$, and $\mathfrak{d}_i^k$ with $0 < k < i$ comes from the natural map $\underline{V}^{\check{\mu}_k}\otimes\underline{V}^{\check{\mu}_{k+1}}\to\underline{V}^{\check{\mu}_k+\check{\mu}_{k+1}}$.

Let $\mathfrak{d}_i$ be the alternating sum of the $\mathfrak{d}_i^j$'s. The commutativity of (3.1) implies that $\mathfrak{d}_{i-1}\circ\mathfrak{d}_i = 0$, i.e.,

$$\dots \to \mathsf{F}(\mathcal{T})^{+(i+1)} \to \mathsf{F}(\mathcal{T})^{+(i)} \to \mathsf{F}(\mathcal{T})^{+(i-1)} \to \dots$$

is a complex. We will denote it by $\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+$.

Proposition 3.12. *Under the assumptions of Proposition 3.10, the natural map* $\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+ \to \mathsf{F}(\mathcal{T})$ *is a quasi-isomorphism.*

Before giving a proof let us consider the following construction. For an element $\check{\lambda}_0 \in \check{\Lambda}^+$, let $\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+_{\geq\check{\lambda}_0}$ be a sub-complex of $\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+$ consisting of terms

$$\mathsf{F}(\mathcal{T})^{+(i)}_{\geq\check{\lambda}_0} := \bigoplus_{\check{\mu}_1,\dots,\check{\mu}_i,\check{\lambda};\check{\lambda}-\check{\lambda}_0\in\check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}-\check{\mu}_1-\dots-\check{\mu}_i})\otimes\underline{V}^{\check{\mu}_1}\otimes\dots\otimes\underline{V}^{\check{\mu}_i}\otimes\mathcal{M}_{\check{\lambda}}.$$

The next assertion holds in the general framework of Sect. 3.3:

Lemma 3.13. *The embedding*

(3.13) $$\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+_{\geq\check{\lambda}_0} \hookrightarrow \mathfrak{C}(\mathsf{F}(\mathcal{T}))^+$$

is a quasi-isomorphism.

Proof. As in (3.5), we have two pairs of maps

$$\mathfrak{C}(\mathsf{F}(\mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}_0}))^+\otimes\underline{V}^{\check{\lambda}_0} \rightrightarrows \mathfrak{C}(\mathsf{F}(\mathcal{T}))^+, \text{ and } \mathfrak{C}(\mathsf{F}(\mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}_0}))^+_{\geq\check{\lambda}_0}\otimes\underline{V}^{\check{\lambda}_0} \rightrightarrows \mathfrak{C}(\mathsf{F}(\mathcal{T}))^+_{\geq\check{\lambda}_0}$$

but it is easy to see that they are tautologically pairwise homotopic.

Let $\mathcal{Y} \subset \mathrm{Fl}^{\check{G}}$ be an affine subvariety, which $\mathcal{T}$ was the direct image from. Let us choose a splitting, denoted by $\mathfrak{s}$, of the surjection

$$\mathcal{O}_{\mathcal{Y}}\otimes\underline{V}^{\check{\lambda}_0} \to \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}_0}|_{\mathcal{Y}}.$$

Then for each $\check{\mu}$ we obtain a splitting

$$\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\mu}-\check{\lambda}_0})\otimes\underline{V}^{\check{\lambda}_0} \overset{\mathfrak{s}}{\hookleftarrow} \Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\mu}}).$$

We obtain a map

$$\mathfrak{C}(\mathsf{F}(\mathcal{T}))^+ \overset{\mathfrak{s}}{\to} \mathfrak{C}(\mathsf{F}(\mathcal{T}\otimes\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{-\check{\lambda}_0}))^+\otimes\underline{V}^{\check{\lambda}_0} \overset{\beta^{\check{\lambda}_0,?}}{\longrightarrow} \mathfrak{C}(\mathsf{F}(\mathcal{T}))^+_{\geq\check{\lambda}_0},$$

which is easily seen to be the inverse to the embedding (3.13), up to homotopy.

$\square$

To prove Proposition 3.12 we shall first consider the case when $\mathcal{C} = \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ and F is the identity functor.

Lemma 3.14. *Proposition 3.12 holds for* $\mathcal{C} = \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ *and* $\mathsf{F} = \mathrm{Id}$.

Proof. In the case under consideration, we will use a shorthand notation $\mathfrak{C}(\mathfrak{T})^+$ (resp., $\mathfrak{C}(\mathfrak{T})^+_{\geq \check{\lambda}_0}$).

For a given $\mathfrak{T}$ it suffices to show that for any $\check{\lambda}_0 \in \check{\Lambda}^+$ the complex of vector spaces $\Gamma\left(\mathrm{Fl}^{\check{G}}, \mathfrak{C}(\mathfrak{T})^+ \otimes \mathcal{L}^{\check{\lambda}_0}_{\mathrm{Fl}^{\check{G}}}\right)$ is quasi-isomorphic to $\Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T} \otimes \mathcal{L}^{\check{\lambda}_0}_{\mathrm{Fl}^{\check{G}}})$.

The above complex is tautologically isomorphic to the complex $\Gamma\left(\mathrm{Fl}^{\check{G}}, \mathfrak{C}(\mathfrak{T} \otimes \mathcal{L}^{\check{\lambda}_0}_{\mathrm{Fl}^{\check{G}}})^+_{\geq \check{\lambda}_0}\right)$, and by Lemma 3.13, the latter is quasi-isomorphic to $\Gamma\left(\mathrm{Fl}^{\check{G}}, \mathfrak{C}(\mathfrak{T} \otimes \mathcal{L}^{\check{\lambda}_0}_{\mathrm{Fl}^{\check{G}}})^+\right)$. Setting $\mathfrak{T}' := \mathfrak{T} \otimes \mathcal{L}^{\check{\lambda}_0}_{\mathrm{Fl}^{\check{G}}}$, we need to verify the exactness of the complex

$$... \to \bigoplus_{\check{\mu}_1,...,\check{\mu}_i,\check{\mu}_{i+1} \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}' \otimes \mathcal{L}^{-\check{\mu}_1 - ... - \check{\mu}_i - \check{\mu}_{i+1}}_{\mathrm{Fl}^{\check{G}}}) \otimes \underline{V}^{\check{\mu}_1} \otimes ... \otimes \underline{V}^{\check{\mu}_i} \otimes \underline{V}^{\check{\mu}_{i+1}} \to ...$$

$$... \to \bigoplus_{\check{\mu}_1,...,\check{\mu}_i \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}' \otimes \mathcal{L}^{-\check{\mu}_1 - ... - \check{\mu}_i}_{\mathrm{Fl}^{\check{G}}}) \otimes \underline{V}^{\check{\mu}_1} \otimes ... \otimes \underline{V}^{\check{\mu}_i} \to ...$$

$$\to \bigoplus_{\check{\mu} \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}' \otimes \mathcal{L}^{-\check{\mu}}_{\mathrm{Fl}^{\check{G}}}) \otimes \underline{V}^{\check{\mu}} \to \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}').$$

However, this complex admits an explicit homotopy

$$\mathsf{h}^i : \bigoplus_{\check{\mu}_1,...,\check{\mu}_i \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}' \otimes \mathcal{L}^{-\check{\mu}_1 - ... - \check{\mu}_i}_{\mathrm{Fl}^{\check{G}}}) \otimes \underline{V}^{\check{\mu}_1} \otimes ... \otimes \underline{V}^{\check{\mu}_i} \to$$

$$\to \bigoplus_{\check{\mu}_1,...,\check{\mu}_i,\check{\mu}_{i+1} \in \check{\Lambda}^+} \Gamma(\mathrm{Fl}^{\check{G}}, \mathfrak{T}' \otimes \mathcal{L}^{-\check{\lambda} - \check{\mu}_1 - ... - \check{\mu}_i - \check{\mu}_{i+1}}_{\mathrm{Fl}^{\check{G}}}) \otimes \underline{V}^{\check{\mu}_1} \otimes ... \otimes \underline{V}^{\check{\mu}_i} \otimes \underline{V}^{\check{\mu}_{i+1}},$$

obtained by taking $\check{\mu}_{i+1} = 0$.

$\square$

Now we are ready to prove Proposition 3.12:

Proof. According to Lemma 3.13, it suffices to show that

$$\mathfrak{C}(\mathsf{F}(\mathfrak{T}))^+_{\geq \check{\lambda}_0} \to \mathfrak{T} \overset{R}{*} \mathcal{W}_{w_0}$$

is a quasi-isomorphism for any $\check{\lambda}_0 \in \check{\Lambda}^+$. We take $\check{\lambda}_0$ to be sufficiently large so that the conclusion of Proposition 3.10 holds.

Then we have:

$$\mathfrak{C}(\mathsf{F}(\mathfrak{T}))^+_{\geq \check{\lambda}_0} \simeq \mathfrak{C}(\mathfrak{T})^+_{\geq \check{\lambda}_0} \overset{R}{*} \mathcal{W}_{w_0},$$

and the assertion follows from the fact that $\mathfrak{C}(\mathfrak{T})^+_{\geq \check{\lambda}_0}$ is quasi-isomorphic to $\mathfrak{T}$, as we have seen above.

$\square$

3.15. **The functor.** We are now going to apply the above general discussion to the category $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$, and construct the sought-after functor

$$\mathsf{E} : D^+\left(\mathrm{QCoh}\left(\mathrm{Fl}^{\check{G}}\right)\right) \to D^+\left(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}\right).$$

Let $R(\check{G})$ denote the left-regular representation of $\check{G}$, and let $\mathcal{F}_{R(\check{G})}$ be the corresponding object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$. However, it is easy to see that $\mathcal{F}_{R(\check{G})}$ is naturally an object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$, and, moreover, as such it is $\check{G}$-equivariant. Furthermore, the assignment

$$\mathcal{F} \mapsto \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_{R(\check{G})}$$

defines an equivalence between $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$ and the category of $\check{G}$-equivariant objects of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$.

Following [FG4], we will denote the above functor also by $\mathrm{Ind}^{\mathrm{Hecke}}$. When viewed as a functor $D(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod} \to D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}$, it is the left adjoint to the tautological forgetful functor.

For an element $\check{\lambda} \in \check{\Lambda}^+$ consider the $\check{G}$-equivariant object of $D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}$ equal to $\mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda},\mathrm{Gr}_G,*})$.

Lemma-Construction 3.16. *For two elements $\check{\lambda}, \check{\mu} \in \check{\Lambda}^+$ there exists a canonical map*

$$(3.14) \qquad \underline{V}^{\check{\mu}} \otimes \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda},\mathrm{Gr}_G,*}) \to \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*}).$$

Proof. We can rewrite

$$\underline{V}^{\check{\mu}} \otimes \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda},\mathrm{Gr}_G,*}) \simeq j_{\check{\lambda},*} \star \mathcal{F}_{V^{\check{\mu}}} \underset{G[[t]]}{\star} \mathcal{F}_{R(\check{G})}$$

and

$$\mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*}) \simeq j_{\check{\lambda},*} \star j_{\check{\mu},\mathrm{Gr}_G,*} \underset{G[[t]]}{\star} \mathcal{F}_{R(\check{G})}.$$

The sought-after morphism comes from the map

$$\mathcal{F}_{V^{\check{\mu}}} \simeq \mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\mu}}_G}} \to j_{\check{\mu},\mathrm{Gr}_G,*}.$$

$\square$

It is easy to see that the morphism of Lemma-Construction 3.16 respects the natural $\check{G}$-equivariant structures on both sides, and the resulting diagrams (3.1) are commutative.

Thus, we find ourselves in the framework of Sections 3.3 and 3.6. We set E to be the resulting functor $D^+\big(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\big) \to D^+\big(D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}\big)$. We let $\mathcal{W}_{w_0}$ denote the object $\mathsf{E}(\mathcal{W}_0) \in D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}$.

3.17. By the results of the previous subsection, for every $\check{\lambda} \in \check{\Lambda}^+$ we have a canonical map

$$(3.15) \qquad \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda},\mathrm{Gr}_G,*}) \to \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}} \overset{R}{*} \mathcal{W}_{w_0} =: \mathsf{E}(\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}}).$$

We claim that assumptions (A) and (B) of Proposition 3.10 hold in our situation. Indeed, assumption (A) follows from [ABBGM], Proposition 2.3.2. The Artinian property of assumption (B) follows from [ABBGM], Corollary 1.3.10. To prove the cohomology vanishing property, we can assume that the object in question is irreducible. Hence, again by [ABBGM], Corollary 1.3.10, it is of the form $\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F})$ for some irreducible $\mathcal{F} \in D(\mathrm{Gr}_G)_{\mathrm{crit}}\,\text{–mod}^I$, and the $\check{B}$-equivariant structure is given by some character $\check{\lambda}' \in \check{\Lambda}$. The vanishing of higher cohomologies follows from the Bott-Borel-Weil theorem for all $\check{\lambda}$ with $\check{\lambda} + \check{\lambda}' \in \check{\Lambda}^+$.

Thus, we obtain that the map (3.15) is an isomorphism for all $\check{\lambda}$ that are deep enough in the dominant chamber. We shall now establish a strengthening of this result:

Proposition 3.18. *The map* (3.15) *is an isomorphism for any $\check{\lambda} \in \check{\Lambda}^+$.*

This proposition implies that our functor E coincides with the one coming from the equivalence of [ABG]. Before giving a proof we need to make a couple of observations.

Since the objects $j_{\check{\lambda},\mathrm{Gr}_G,*}$ are I-equivariant, by construction, the functor E factors as

$$D^+\big(\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})\big) \to D^+\big(D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}\big)^I \to D^+\big(D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}\big).$$

In particular, its image belongs to $D^+\big(D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}\big)^{I^0}$, the latter being a full triangulated subcategory of $D^+\big(D(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\,\text{–mod}\big)$.

Proposition 3.19. *For* $\check{\lambda} \in \check{\Lambda}^+$ *and* $\mathfrak{T} \in D^+\big(\mathrm{QCoh}\big(\mathrm{Fl}^{\check{G}}\big)\big)$ *there exists a functorial isomorphism*

$$\mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathfrak{T}) \simeq j_{\check{\lambda},*} \star \mathsf{E}(\mathfrak{T}) \in D^+\big(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \,\mathrm{-mod}\big)^I$$

Proof. By the construction of the functor E, it suffices to show that $j_{\check{\lambda},*} \star \mathcal{W}_{w_0}$ is isomorphic to $\mathcal{L}_{\mathrm{Fl}_{w_0}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{W}_{w_0}$ as a $\check{B}$-equivariant object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \,\mathrm{-mod}^I$.

Since the functor $j_{\check{\lambda},*}\star?$ is right exact, $j_{\check{\lambda},*} \star \mathcal{W}_{w_0}$ equals the co-equalizer of the maps

$$\underset{\check{\mu},\check{\lambda}'}{\oplus} \underline{V}^{\check{\mu}} \otimes \mathcal{L}_{\mathrm{Fl}_{w_0}^{\check{G}}}^{-\check{\lambda}'-\check{\mu}} \otimes j_{\check{\lambda},*} \star \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}',\mathrm{Gr}_G,*}) \rightrightarrows \underset{\check{\lambda}''}{\oplus} \mathcal{L}_{\mathrm{Fl}_{w_0}^{\check{G}}}^{-\check{\lambda}''} \otimes \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}'',\mathrm{Gr}_G,*}).$$

Recall that $j_{\check{\lambda},*} \star \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}',\mathrm{Gr}_G,*}) \simeq \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}+\check{\lambda}',\mathrm{Gr}_G,*})$. Therefore, we obtain that the assertion of the proposition holds from Lemma 3.13.

$\square$

We are now ready to prove Proposition 3.18.

Proof. Let $\check{\lambda}$ be as in the proposition, and let $\check{\lambda}'$ be large so that the map

$$\mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}+\check{\lambda}',\mathrm{Gr}_G,*}) \to \mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}+\check{\lambda}'})$$

is an isomorphism. Let us apply the functor $j_{\check{\lambda}',*}\star?$ to the two sides of (3.15). We obtain a commutative diagram

$$
\begin{array}{ccc}
j_{\check{\lambda}',*} \star \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda},\mathrm{Gr}_G,*}) & \xrightarrow{\ (3.15)\ } & j_{\check{\lambda}',*} \star \mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}) \\
\Big\downarrow & & \text{Proposition 3.19} \Big\downarrow \\
\mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}+\check{\lambda}',\mathrm{Gr}_G,*}) & \xrightarrow{\ (3.15)\ } & \mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}+\check{\lambda}'}),
\end{array}
$$

and our assertion follows from the fact that the functor $j_{\check{\lambda}',*}\star?$ is a self-equivalence of $D^+(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \,\mathrm{-mod})^I$

$\square$

4. Identification of Wakimoto modules

The goal of this section is to formulate a result, Theorem 4.8, that will describe Wakimoto modules in terms of the equivalence (0.2) and the functor E.

4.1. **The category.** We begin by spelling out the constructions of the previous section in a relative situation, namely, over the scheme $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) = \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$. Recall that over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ there exists a canonical $\check{G}$-torsor, denoted $\mathcal{P}_{\check{G},\mathfrak{Z}}$, that corresponds to the tautological $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$ over $\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ under the isomorphism $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ induced by $\mathrm{map}_{\mathrm{alg}} = \mathrm{map}_{\mathrm{geom}}$. We will denote by $\mathcal{P}_{\check{B},\mathfrak{Z}}$ its reduction to $\check{B}$, corresponding to the oper structure on $\mathcal{P}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$. For $V \in \mathrm{Rep}(\check{G})$, recall that $\mathcal{V}_{\mathfrak{Z}}$ denotes the associated vector bundle over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$.

Following [FG4], we introduce the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}} \,\mathrm{-mod}$ as follows. Its objects are the data of $(\mathcal{F}, \alpha_V, \forall\, V \in \mathrm{Rep}(\check{G}))$, where $\mathcal{F}$ is an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}} \,\mathrm{-mod}$, endowed with an action of the algebra $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ by endomorphisms, and $\alpha_V, V \in \mathrm{Rep}(\check{G})$, are isomorphisms of D-modules

$$\alpha_V : \mathcal{F} \star \mathcal{F}_V \xrightarrow{\ \sim\ } \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F},$$

compatible with the action of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ on both sides, and such that the following two conditions are satisfied:

- If V is the trivial representations $\mathbb{C}$, then the morphism α_V is the identity map.
- For $V, W \in \mathrm{Rep}(\check{G})$ and $U := V \otimes W$, the diagram

$$
\begin{array}{ccc}
(\mathcal{F} \star \mathcal{F}_V) \star \mathcal{F}_W & \overset{\sim}{\longrightarrow} & \mathcal{F} \star \mathcal{F}_U \\
{\scriptstyle \alpha_V \star \mathrm{id}_{\mathcal{F}_W}} \downarrow & & \downarrow {\scriptstyle \alpha_U} \\
(\mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F}) \star \mathcal{F}_W & & \mathcal{U}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F} \\
\sim \downarrow & & \downarrow \sim \\
\mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} (\mathcal{F} \star \mathcal{F}_W) & \xrightarrow{\mathrm{id}_{\mathcal{V}_{\mathfrak{Z}}} \otimes \alpha_W} & \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{W}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F}
\end{array}
$$

is commutative.

Morphisms in this category between $(\mathcal{F}, \alpha_V)$ and $(\mathcal{F}', \alpha'_V)$ are maps of D-modules $\phi : \mathcal{F} \to \mathcal{F}'$ that are compatible with the actions of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ on both sides, and such that

$$
(\mathrm{id}_{\mathcal{V}_{\mathfrak{Z}}} \otimes \phi) \circ \alpha_V = \alpha'_V \circ (\phi \star \mathrm{id}_{\mathcal{F}_V}).
$$

Note that the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ can be (non-canonically) trivialized. A choice of such a trivialization defines an equivalence between $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}$ –mod and the category consisting of objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}$ –mod, endowed with an action of the algebra $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$.

Let $R(\check{G})^{\mathfrak{Z}}$ denote the direct image of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ onto $\mathrm{pt}/\check{G}$, regarded as an object of $\mathrm{Rep}(\check{G})$, endowed with a commuting action of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$. Let $\mathcal{F}_{R(\check{G})^{\mathfrak{Z}}}$ be the corresponding object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ –mod.

As in the case of $\mathcal{F}_{R(\check{G})}$, one shows that $\mathcal{F}_{R(\check{G})^{\mathfrak{Z}}}$ is naturally an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ –mod. Moreover, the assignment

$$
\mathcal{F} \mapsto \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_{R(\check{G})^{\mathfrak{Z}}}
$$

defines a functor $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$ –mod $\to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ –mod, denoted, $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}$, which is the left adjoint to the tautological forgetful functor.

4.2. **The functor to $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod$_{\mathrm{reg}}$.** Let us now recall the definition of the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$: $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ –mod $\to \widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod$_{\mathrm{reg}}$, following [FG2] and [FG4].

Consider the groupoid

$$
(4.1) \qquad \mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}} : \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}),
$$

where the morphism $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathrm{pt}/\check{G}$ corresponds to the $\check{G}$-bundle $\mathcal{P}_{\check{G},\mathfrak{Z}}$ on $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$. Let $\mathbf{1}_{\mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}}$ denote the unit section $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$. As shown in *loc. cit.*, for every object $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ –mod, the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module $\Gamma(\mathrm{Gr}_G, \mathcal{F}^H)$ carries a natural action of the algebra $\mathrm{Fun}(\mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}})$ by endomorphisms. We then define the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ by

$$
\mathcal{F}^H \mapsto \Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \underset{\mathrm{Fun}(\mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}), \mathbf{1}^*_{\mathsf{Isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}.
$$

For $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$ –mod we have a natural isomorphism:

$$
\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F}).
$$

In [FG2] we conjectured that the functor $\Gamma^{\text{Hecke}_{\mathfrak{z}}}$ is exact and defines an equivalence of categories $\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{z}}}$ –mod and $\widehat{\mathfrak{g}}_{\text{crit}}$ –mod$_{\text{reg}}$. In [FG4], Theorem 1.7, we proved that the corresponding functor

$$(4.2) \qquad \mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{z}}} -\mathrm{mod}^{I^0} \to \widehat{\mathfrak{g}}_{\text{crit}} -\mathrm{mod}_{\text{reg}}^{I^0},$$

between the I^0-equivariant categories, is indeed an equivalence of categories.

4.3. **The functor $\mathsf{E}^{\mathfrak{z}}$.** Let $\check{G}_{\mathfrak{z}}$ be the group-scheme over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\text{reg}})$ equal to the adjoint twist of $\check{G}$ by means of $\mathcal{P}_{\check{G},\mathfrak{z}}$. By construction, $\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{z}}}$ –mod, viewed as a category over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\text{reg}})$ carries an action of $\check{G}_{\mathfrak{z}}$ in the same way as $\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}}$ –mod carried an action of $\check{G}$.

Let $\check{B}_{\mathfrak{z}}$ denote the group-subscheme of $\check{G}_{\mathfrak{z}}$, corresponding to the reduction $\mathcal{P}_{\check{B},\mathfrak{z}}$ of $\mathcal{P}_{\check{G},\mathfrak{z}}$ to $\check{B}$. Let $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$ be the scheme over $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\text{reg}})$, classifying reductions of $\mathcal{P}_{\check{G},\mathfrak{z}}$ to $\check{B}^- \subset \check{G}$, i.e.,

$$\mathrm{Fl}_{\mathfrak{z}}^{\check{G}} = \mathcal{P}_{\check{G},\mathfrak{z}} \overset{\check{G}}{\times} \mathrm{Fl}^{\check{G}}.$$

The group-scheme $\check{G}_{\mathfrak{z}}$ acts naturally on $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$.

The $\check{B}_{\mathfrak{z}}$-orbits on $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$ are in a natural bijection with the elements of the Weyl group; for $w \in W$ we will denote by $\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}} \subset \mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$ the Schubert stratum, corresponding to the coset $\check{B} \cdot w^{-1} \cdot \check{B}^-$.

For $w = 1$, this is an open subscheme of $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$. For $w = w_0$, the subscheme $\mathrm{Fl}_{w_0,\mathfrak{z}}^{\check{G}} \subset \mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$ is the section of the natural projection

$$p : \mathrm{Fl}_{\mathfrak{z}}^{\check{G}} \to \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}^{\text{reg}}),$$

corresponding to the reduction $\mathcal{P}_{\check{B},\mathfrak{z}}$ of $\mathcal{P}_{\check{G},\mathfrak{z}}$. The stabilizer of $\mathrm{Fl}_{w_0,\mathfrak{z}}^{\check{G}}$ in $\check{G}_{\mathfrak{z}}$ is, by definition, $\check{B}_{\mathfrak{z}}$.

For $\check{\lambda} \in \check{\Lambda}^+$, let $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}}^{\check{\lambda}}$ denote the corresponding $\check{G}_{\mathfrak{z}}$-equivariant line bundle on $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$, normalized so that

$$p_*(\mathcal{L}_{\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}}^{\check{\lambda}}) \simeq \mathcal{V}_{\mathfrak{z}}^{\check{\lambda}}.$$

As in the case of $\mathrm{Fl}^{\check{G}}$ and $\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}}$ –mod, we construct a functor

$$(4.3) \qquad \mathsf{E}^{\mathfrak{z}} : D^+\big(\mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{z}}^{\check{G}})\big) \to D^+\big(\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{z}}} -\mathrm{mod}\big).$$

It is obtained as the convolution $\mathsf{E}^{\mathfrak{z}}(\mathcal{T}) = \mathcal{T} \overset{R}{*} \mathcal{W}_{w_0}^{\mathfrak{z}}$, where $\mathcal{W}_{w_0}^{\mathfrak{z}}$ is a $\check{B}_{\mathfrak{z}}$-equivariant object of $\mathrm{D}(\mathrm{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{z}}}$ –mod that can be recovered as $\mathsf{E}^{\mathfrak{z}}(\mathcal{O}_{\mathrm{Fl}_{w_0,\mathfrak{z}}^{\check{G}}})$.

Explicitly, for $\mathcal{T}$ being the direct image of a quasi-coherent sheaf on an affine locally closed subscheme of $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$ (in particular, for $\mathcal{T} = \mathcal{O}_{\mathrm{Fl}_{w_0,\mathfrak{z}}^{\check{G}}}$), the object $\mathsf{E}^{\mathfrak{z}}(\mathcal{T})$ is defined as follows:

Set

$$\mathsf{E}^{\mathfrak{z}}(\mathcal{T})^+ := \underset{\check{\lambda}\in\check{\Lambda}^+}{\oplus} p_*(\mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}) \underset{\mathfrak{z}_{\mathfrak{g}}^{\text{reg}}}{\otimes} \mathrm{Ind}^{\text{Hecke}_{\mathfrak{z}}}(j_{\check{\lambda},\mathrm{Gr}_G,*})$$

and

$$\mathsf{E}^{\mathfrak{z}}(\mathcal{T})^{++} := \underset{\check{\mu},\check{\lambda}\in\check{\Lambda}^+}{\oplus} \mathcal{V}_{\mathfrak{z}}^{\check{\mu}} \underset{\mathfrak{z}_{\mathfrak{g}}^{\text{reg}}}{\otimes} p_*(\mathcal{T} \otimes \mathcal{L}_{\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}) \underset{\mathfrak{z}_{\mathfrak{g}}^{\text{reg}}}{\otimes} \mathrm{Ind}^{\text{Hecke}_{\mathfrak{z}}}(j_{\check{\lambda},\mathrm{Gr}_G,*})$$

Then $\mathsf{E}^{\mathfrak{z}}(\mathcal{T})$ is, by definition, the co-equalizer of the map

$$\mathsf{E}^{\mathfrak{z}}(\mathcal{T})^{++} \rightrightarrows \mathsf{E}^{\mathfrak{z}}(\mathcal{T})^+.$$

We now set $\mathcal{W}^3_{w_0} = \mathsf{E}^3(\mathcal{O}_{\mathrm{Fl}^{\check{G}}_{w_0,3}})$. It follows from the construction that $\mathcal{W}^3_{w_0}$ is a $\check{B}_3$-equivariant object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}}$ –mod. Having defined $\mathcal{W}^3_{w_0}$, we define the functor (4.3) as follows:

We have a functor

$$\mathrm{act}^*_3 : \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}} \text{–mod} \to \mathrm{QCoh}(\check{G}_3) \underset{3^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}} \text{–mod}$$

corresponding to the action of $\check{G}_3$ on $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}}$ –mod. Let $\mathcal{T}$ be a quasi-coherent sheaf on $\mathrm{Fl}^{\check{G}}_3$ and $\widetilde{\mathcal{T}}$ the corresponding $\check{B}_3$-equivariant quasi-coherent sheaf on $\check{G}_3$. We set

$$(4.4) \qquad \mathsf{E}^3(\mathcal{T}) := \mathcal{T} \overset{R}{*} \mathcal{W}^3_{w_0} := R \operatorname{Inv}\left(\check{B}_3, \widetilde{\mathcal{T}} \underset{\mathrm{Fun}(\check{G}_3)}{\otimes} \mathrm{act}^*_3(\mathcal{W}^3_{w_0}) \right),$$

where $R \operatorname{Inv}\left(\check{B}_3, ? \right)$ denotes the derived functor of $\check{B}_3$-invariants.

The object of $D^+\left(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}\right)$, underlying $\mathsf{E}^3(\mathcal{T})$, can be explicitly written as follows:

$$\mathsf{E}^3(\mathcal{T}) = R \operatorname{Inv}\left(\check{B}_3, \widetilde{\mathcal{T}} \underset{3^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{W}^3_{w_0} \right).$$

The next lemma follows from Proposition 3.18:

Lemma 4.4. *For $\check{\lambda} \in \check{\Lambda}^+$ we have:*

$$\mathsf{E}^3(\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_3}) \simeq \mathrm{Ind}^{\mathrm{Hecke}_3}(j_{\check{\lambda},\mathrm{Gr}_G,*}).$$

Let now $\check{\lambda}$ and $\check{\mu}$ be two elements of $\check{\Lambda}^+$. As in Lemma-Construction 3.16 we have a canonical map

$$(4.5) \qquad \mathcal{V}^{\check{\mu}}_3 \underset{3^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathrm{Ind}^{\mathrm{Hecke}_3}(j_{\check{\lambda},\mathrm{Gr}_G,*}) \to \mathrm{Ind}^{\mathrm{Hecke}_3}(j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*}),$$

and a commutative diagram:

$$\begin{array}{ccc}
\mathsf{E}^3(p^*(\mathcal{V}^{\check{\mu}}_3) \otimes \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_3}) & \longrightarrow & \mathcal{V}^{\check{\mu}}_3 \underset{3^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathrm{Ind}^{\mathrm{Hecke}_3}(j_{\check{\lambda},\mathrm{Gr}_G,*}) \\
\downarrow & & \downarrow \\
\mathsf{E}^3(\mathcal{L}^{\check{\lambda}+\check{\mu}}_{\mathrm{Fl}^{\check{G}}_3}) & \longrightarrow & \mathrm{Ind}^{\mathrm{Hecke}_3}(j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*}).
\end{array}$$

Finally, as in the case of E, the functor E^3 factors as

$$D^+\left(\mathrm{QCoh}\left(\mathrm{Fl}^{\check{G}}_3\right)\right) \to D^+(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}}\text{–mod})^I \to D^+(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}}\text{–mod}),$$

and as in Proposition 3.19, we have:

$$(4.6) \qquad\qquad j_{\check{\lambda},*} \star \mathsf{E}^3(\mathcal{T}) \simeq \mathsf{E}^3(\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_3} \otimes \mathcal{T}).$$

4.5. **The functor G.** Let us denote by G the functor

$$\Gamma^{\mathrm{Hecke}_3} \circ \mathsf{E}^3 : D^+\left(\mathrm{QCoh}\left(\mathrm{Fl}^{\check{G}}_3\right)\right) \to D^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}})^{I^0}.$$

The goal of the rest of this paper is to describe the relationship between this functor and Wakimoto modules.

4.6. $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$ **and Miura opers.** Recall the D-scheme $\mathrm{MOp}_{\check{\mathfrak{g}},X}$ of Miura opers, and let $\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ be the corresponding scheme, attached to the formal disc. By the construction of $\mathrm{map}_{\mathrm{geom}}$, it identifies the $\check{G}$-torsors $\mathcal{P}_{\check{G},\mathfrak{Z}}$ over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ and $\mathcal{P}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$ over $\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$. Hence, we have an isomorphism

$$(4.7) \qquad \begin{array}{ccc} \mathrm{Fl}_{\mathfrak{Z}}^{\check{G}} & \xrightarrow{\ \sim\ } & \mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}} \\ p\downarrow & & \downarrow \\ \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}\ } & \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}. \end{array}$$

For every $w \in W$, let $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}} \subset \mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ be the corresponding Schubert cell; under the isomorphism (4.7), $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}$ goes over to $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$. Let $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{th},\mathrm{reg}}$ (resp., $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$) denote the formal neighborhood of $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}$ in $\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ (resp., of $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$ in $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$).

Recall the D-scheme $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$, and let $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)$ (resp., $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$) be the scheme (resp., ind-scheme) of its sections over the formal (resp., punctured) disc around x. Recall the isomorphism (2.3) $\mathrm{Conn}_{\check{H}}(\omega_X^\rho) \simeq \mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X}$. Taking the fibers at $x \in X$ we obtain an isomorphism

$$\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho) \simeq \mathrm{MOp}_{\check{\mathfrak{g}}}^{1,\mathrm{reg}}.$$

From [FG2], Sect. 3.6, we obtain that the above isomorphism generalizes to the following canonical isomorphism

$$(4.8) \qquad \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} \simeq \underset{w\in W}{\bigsqcup} \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{th},\mathrm{reg}}.$$

From (1.11), we obtain a commutative diagram of ind-schemes

$$\begin{array}{ccc} \mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}^M\ } & \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \\ \varphi\downarrow & & \mathrm{MT}\downarrow \\ \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}\ } & \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times). \end{array}$$

Combing this with (4.8), we obtain an isomorphism

$$(4.9) \qquad \mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) \underset{\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})}{\times} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \simeq \underset{w\in W}{\bigsqcup} \mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}},$$

which will play a crucial role for the rest of this paper.

4.7. **Relation to Wakimoto modules.** Recall from Sect. 2.2 the functor $\mathbb{W} : \widehat{\mathfrak{H}}_{\mathrm{crit}}\,\text{--mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\,\text{--mod}$. In particular, we obtain a functor

$$\mathrm{QCoh}\Big(\mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) \underset{\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})}{\times} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})\Big) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\,\text{--mod}_{\mathrm{reg}}.$$

Using (4.9), for each element w of the Weyl group, we obtain a functor

$$(4.10) \qquad {}_w\mathbb{W} : \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\,\text{--mod}_{\mathrm{reg}}.$$

The direct image functor identifies $\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ with a full subcategory of $\mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}})$. Our main result compares the functors ${}_w\mathbb{W}$ and $\mathsf{G}|_{\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})}$:

Theorem 4.8. *For* $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ *there exists a canonical isomorphism:*

$$_w\mathbb{W}(\mathcal{T}) \simeq \mathsf{G}\Big(\mathcal{T} \underset{\mathcal{O}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}}{\otimes} \mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{\check{\rho} - w(\check{\rho})}|_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}\Big).$$

4.9. **Some particular cases.** For a weight $\mu \in \mathfrak{h}^*$, let $\mathbb{W}_{\mathrm{crit},\mu}$ be the corresponding Wakimoto module (see Sect. 2.2). By definition, it is induced from the $\widehat{\mathfrak{H}}_{\mathrm{crit}}$-module $\mathrm{Fun}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(\mu)})$.

Its support over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is contained in $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{RS},\varpi(-w_0(\mu)+\rho)})$. Let us take $\mu = w(\rho) - \rho$; then $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{RS},\varpi(-w_0(\mu)+\rho)}) = \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}})$.

We introduce the module

$$\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} := \mathbb{W}_{\mathrm{crit},w(\rho)-\rho} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}.$$

This is the maximal quotient of $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}$ that belongs to the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathrm{mod}_{\mathrm{reg}}$. In the particular case when $w = 1$, the module $\mathbb{W}_{\mathrm{crit},0}$ is itself supported over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$, and so $\mathbb{W}_{\mathrm{crit},0} \to \mathbb{W}_{\mathrm{crit},1,\mathrm{reg}}$ is an isomorphism.

We would like now to apply Theorem 4.8 and describe the above modules in terms of objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–}\mathrm{mod}$.

For every $\lambda \in \mathfrak{h}^*$ we can consider the D-scheme $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)^{\mathrm{RS},\lambda}$, whose restriction to $X - x$ is isomorphic to $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)$, and whose fiber at x identifies with $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)^{\mathrm{RS},\lambda}$, where the latter is as in [FG2], Sect. 3.5. By definition, $\mathrm{Conn}_{\check{H}}(\omega_X^\rho)^{\mathrm{RS},\lambda}$ is the scheme of connections on the $\check{H}$-bundle ω_X^ρ that have a pole of order 1 at x with residue λ.

Let us take $\lambda = \rho - w(\rho)$. By [FG2], Sect. 3.6, the morphism

$$\mathrm{Conn}_{\check{H}}(\omega_X^\rho) \to \mathrm{MOp}_{\check{\mathfrak{g}},X},$$

given by (2.3), composed with the tautological embedding $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen},X} \hookrightarrow \mathrm{MOp}_{\check{\mathfrak{g}},X}$ extends to a map of D-schemes

$$(4.11) \qquad \mathrm{Conn}_{\check{H}}(\omega_X^\rho)^{\mathrm{RS},\rho-w(\rho)} \underset{\mathrm{Op}_{\check{\mathfrak{g}},X}^{\mathrm{nilp}}}{\times} \mathrm{Op}_{\check{\mathfrak{g}},X}^{\mathrm{reg}} \to \mathrm{MOp}_{\check{\mathfrak{g}},X},$$

where $\mathrm{Op}_{\check{\mathfrak{g}},X}^{\mathrm{nilp}}$ is the D-scheme of opers with a nilpotent singularity at x.

The resulting map on the level of fibers fits into a commutative diagram

$$(4.12)$$

$$\begin{array}{ccccc}
\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)^{\mathrm{RS},\rho-w(\rho)} \underset{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{nilp}}}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} & \xrightarrow{\sim} & \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}} & \longrightarrow & \mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}} \\
\downarrow & & \downarrow & & \\
\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} & \xrightarrow{(4.8)} & \underset{w \in W}{\sqcup} \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{th},\mathrm{reg}}. & &
\end{array}$$

The left portion of the upper horizontal arrow is an isomorphism, and hence it identifies $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)^{\mathrm{RS},\rho-w(\rho)} \underset{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{nilp}}}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ with the reduced scheme of a connected component of $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$.

Combining this with the isomorphism $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \simeq \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)^{\mathrm{RS},\rho-w(\rho)}$, induced by $\mathrm{map}_{\mathrm{geom}}^M$, we obtain an isomorphism:

$$(4.13) \qquad \mathrm{Spec}\Big(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\Big) \simeq \mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}},$$

where regular functions on the (affine) scheme on the LHS is, by definition, the $\widehat{\mathfrak{H}}_{\mathrm{crit}}$-module that $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}}$ is induced from. In other words,

$$\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \simeq {}_w\mathbb{W}(\mathcal{O}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}).$$

Let us introduce a short-hand notation $\mathcal{W}^{\mathfrak{Z}}_w := \mathsf{E}^{\mathfrak{Z}}(\mathcal{O}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}})$. As we will see below, for every $\check{\lambda} \in \check{\Lambda}$ and $w \in W$, the restriction of the line bundle $\mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}}$ to $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}$ is constant and isomorphic to $\mathfrak{l}^{\check{\lambda}}_{w(\rho)} \otimes \mathcal{O}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}$, where $\mathfrak{l}^{\check{\lambda}}_{w(\rho)}$ is as in Proposition 2.4. Hence, we obtain:

Theorem 4.10. *There exists a canonical isomorphism*

$$\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \mathfrak{l}^{w(\check{\rho})-\check{\rho}}_{w(\rho)} \simeq \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{W}^{\mathfrak{Z}}_w).$$

We should note that a particular case of Theorem 4.10, namely, for $w = w_0$ has been established in [FG4], and it was a key calculation on which the proof of the main result of *loc. cit.* was based. The method of proof of Theorem 4.10 for a general w presented below will be quite different.

Taking $w = 1$, we obtain the following description of the "main" Wakimoto module:

Theorem 4.11. *There exists a canonical isomorphism*

$$\mathbb{W}_{\mathrm{crit},0} \simeq \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{W}^{\mathfrak{Z}}_1).$$

Our strategy for the proof of Theorem 4.8 will be as follows. First, we will prove Theorem 4.11; this will be done in Sect. 5 by a rather explicit argument. Then, in Sect. 6, we will prove Theorem 4.10. In Sect. 7 we will prove a weakened version of Theorem 4.8: namely, we will show that

$$ {}_w\mathbb{W}(\mathcal{T}) \simeq \mathsf{G}\Big(\mathcal{L}^{\mathrm{twist}}_w \underset{\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})}{\otimes} \mathcal{T}\Big),$$

where $\mathcal{L}^{\mathrm{twist}}_w$ is a certain *bi-module* over the topological algebra $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$.

Finally, in Sect. 9, we will show that the left and right actions of $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$ on $\mathcal{L}^{\mathrm{twist}}_w$ coincide, and that it is an invertible sheaf canonically isomorphic to $\mathcal{L}^{\check{\rho}-w(\check{\rho})}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}}|_{\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}}$.

4.12. **A BGG type resolution.** As an application of Theorem 4.8, we construct a BGG type resolution of the vacuum module $\mathbb{V}_{\mathrm{crit}}$. For an element $w \in W$ let Dist_w denote the $\mathcal{O}$-module on $\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}$ underlying the left D-module of distributions on the Schubert cell $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}$. This $\mathcal{O}$-module can be naturally thought of as an object of $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$.

The Cousin-Grothendieck resolution of the structure sheaf of $\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}$ by means of Dist_w, combined with Lemma 4.4, yields the following:

Corollary 4.13. *There exists a right resolution $C^\bullet$ of $\mathbb{V}_{\mathrm{crit}}$, whose kth term C^k is isomorphic to*

$$\bigoplus_{w \in W, \ell(w)=k} {}_w\mathbb{W}\big(\mathcal{L}^{w(\check{\rho})-\check{\rho}}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}} \underset{\mathcal{O}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}}}{\otimes} \mathrm{Dist}_w\big).$$

In Sect. 10 a more explicit realization of the modules ${}_w\mathbb{W}\big(\mathrm{Dist}_w \underset{\mathcal{O}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}}}{\otimes} \mathcal{L}^{w(\check{\rho})-\check{\rho}}_{\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}}}\big)$, involved in this resolution, will be obtained. In addition, we will make contact with a conjecture of [FF3].

5. Geometric realization of the Wakimoto module $\mathbb{W}_{\mathrm{crit},0}$

As was mentioned above, the goal of this section is to prove Theorem 4.11, which is a particular case of Theorem 4.8. First, we will give a more explicit description of the objects $\mathcal{W}_w^{\mathfrak{Z}}$, which is valid for any $w \in W$.

5.1. Explicit construction of $\mathcal{W}_w^{\mathfrak{Z}}$.

By definition, the object $\mathcal{W}_w^{\mathfrak{Z}} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ is obtained by applying the functor $\mathsf{E}^{\mathfrak{Z}}$ to $\mathcal{O}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$. Since $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$ is affine, $\mathsf{E}^{\mathfrak{Z}}(\mathcal{O}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}})$ can be explicitly described as a quotient as in Sect. 4.3.

Let us denote the restriction $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$ by $\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}$. In what follows we will not distinguish in the notation between quasi-coherent sheaves on this scheme and their global sections, thought of as $\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$-modules.

Set

$$\mathcal{W}_w^{\mathfrak{Z}+} := \bigoplus_{\check{\lambda} \in \check{\Lambda}^+} j_{\check{\lambda},*} \star \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_{1,\mathrm{Gr}_G}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}},$$

and

$$\mathcal{W}_w^{\mathfrak{Z}++} := \bigoplus_{\check{\mu},\check{\lambda} \in \check{\Lambda}^+} j_{\check{\lambda},*} \star \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_{1,\mathrm{Gr}_G}) \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check{\mu}}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}-\check{\mu}}.$$

Using the maps

$$(5.1) \qquad j_{\check{\lambda},*} \star \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_{1,\mathrm{Gr}_G}) \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check{\mu}}} \to j_{\check{\lambda}+\check{\mu},*} \star \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_{1,\mathrm{Gr}_G})$$

and

$$(5.2) \qquad \kappa^{-,\check{\mu},w} : \mathcal{V}_{\mathfrak{Z}}^{\check{\mu}} \simeq \Gamma(\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}, \mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{\check{\mu}}) \to \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\mu}},$$

we obtain two maps $\mathcal{W}_w^{\mathfrak{Z}++} \to \mathcal{W}_w^{\mathfrak{Z}+}$, and $\mathcal{W}_w^{\mathfrak{Z}}$ is, by definition, their co-equalizer.

By construction, the algebra $\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$ acts on $\mathcal{W}_w^{\mathfrak{Z}}$ by endomorphisms. Moreover, for $\check{\lambda} \in \check{\Lambda}^+$ we have a canonical map

$$(5.3) \qquad j_{\check{\mu},*} \star \mathcal{W}_w^{\mathfrak{Z}} \to \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\mu}} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})}{\otimes} \mathcal{W}_w^{\mathfrak{Z}}.$$

By (4.6), this map is in fact an isomorphism.

5.2. The universal property.

Let $\mathcal{M}$ be an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}^I$, endowed with an action of $\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$, compatible with the action of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$, and a system of morphisms

$$(5.4) \qquad j_{\check{\mu},*} \star \mathcal{M} \to \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\mu}} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})}{\otimes} \mathcal{M},$$

for $\check{\mu} \in \check{\Lambda}^+$, such that the diagrams

$$
\begin{array}{ccc}
j_{\check{\mu}_1,*} \star j_{\check{\mu}_2,*} \star \mathcal{M} & \longrightarrow & j_{\check{\mu}_1,*} \star (\mathcal{L}^{\check{\mu}_2}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} \mathcal{M}) \\
\sim \downarrow & & \downarrow \\
j_{\check{\mu}_1+\check{\mu}_2,*} \star \mathcal{M} & & \mathcal{L}^{\check{\mu}_2}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} (j_{\check{\mu}_1,*} \star \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathcal{L}^{\check{\mu}_1+\check{\mu}_2}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} \mathcal{M} & \overset{\sim}{\longrightarrow} & \mathcal{L}^{\check{\mu}_2}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} \mathcal{L}^{\check{\mu}_1}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} \mathcal{M}
\end{array}
$$

are commutative for all $\check{\mu}_1, \check{\mu}_2 \in \check{\Lambda}^+$.

Lemma 5.3. *Under the above conditions, specifying a map*

$$
\Gamma^{\mathrm{Hecke}_3}(\mathrm{Gr}_G, \mathcal{W}^3_w) \to \mathcal{M},
$$

which is compatible with the action of $\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)$, *and which intertwines the morphisms (5.3) and (5.4), is equivalent to specifying a map* $\mathbb{V}_{\mathrm{crit}} \to \mathcal{M}$ *such that the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathcal{F}_{V^{\check{\lambda}}} \star \mathbb{V}_{\mathrm{crit}} \overset{\sim}{\longrightarrow} \mathcal{V}^{\check{\lambda}}_3 \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathbb{V}_{\mathrm{crit}} \longrightarrow & & \mathcal{V}^{\check{\lambda}}_3 \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M} \\
\downarrow & & \kappa^{-,\check{\lambda},w} \downarrow \\
j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \longrightarrow j_{\check{\lambda},*} \star \mathcal{M} \longrightarrow & & \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w,3}} \underset{\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{w,3}\right)}{\otimes} \mathcal{M}.
\end{array}
$$

5.4. Let us now specialize to the case $w = 1$, and let us take $\mathcal{M} = \mathbb{W}_{\mathrm{crit},0}$. We define an action of $\mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{1,3}\right)$ on it via the tautological action of $\mathfrak{H}^{\mathrm{reg}}_{\mathrm{crit}}$ on $\mathbb{W}_{\mathrm{crit},0}$ and the isomorphism

$$
\mathrm{Spec}(\mathfrak{H}^{\mathrm{reg}}_{\mathrm{crit}}) \simeq \mathrm{Fl}^{\check{G}}_{1,3}.
$$

The isomorphism (5.4) is given by Proposition 2.4 and the identification

$$
(5.5) \qquad\qquad\qquad\qquad \mathcal{L}^{\check{\lambda}}_{\mathfrak{H},0} \simeq \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{1,3}},
$$

which follows from the construction of $\mathrm{map}^M_{\mathrm{geom}}$ in Sect. 2.7.

The fact that the conditions of Lemma 5.3 are satisfied, follows from the definitions.

5.5. Thus, we have constructed a map

$$
(5.6) \qquad\qquad\qquad\qquad \Gamma^{\mathrm{Hecke}_3}(\mathrm{Gr}_G, \mathcal{W}^3_1) \to \mathbb{W}_{\mathrm{crit},0},
$$

and our present goal is to show that it is an isomorphism.

First, let us show that this map is injective. We have a natural map

$$
\mathrm{Ind}^{\mathrm{Hecke}_3}(\delta_{1,\mathrm{Gr}_G}) \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{1,3}\right) \to \mathcal{W}^3_1,
$$

where $\mathrm{Ind}^{\mathrm{Hecke}_3}$ is as in [FG4], Sect. 2.5. Moreover, by using the same argument as in [ABBGM], Proposition 3.2.5, we find that the object $\mathcal{W}^3_1 \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_3}_{\mathrm{crit}}\mathrm{-mod}^I$ does not have sub-objects that do not intersect the image of $\mathrm{Ind}^{\mathrm{Hecke}_3}(\delta_{1,\mathrm{Gr}_G}) \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathrm{Fun}\left(\mathrm{Fl}^{\check{G}}_{1,3}\right)$.

Therefore, since the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ is an equivalence, to prove the injectivity of (5.6), it suffices to show that the composition

$$\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}\big(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}\big) \simeq \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}\Big(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_{1,\mathrm{Gr}_G}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}\big(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}\big)\Big) \to$$

$$\to \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}\big(\mathrm{Gr}_G, \mathcal{W}_1^{\mathfrak{Z}}\big) \to \mathbb{W}_{\mathrm{crit},0}.$$

However, by construction, this composition is the canonical map

$$\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathfrak{g}}^{\mathrm{reg}} \simeq (\mathbb{W}_{\mathrm{crit},0})^{G[[t]]} \hookrightarrow \mathbb{W}_{\mathrm{crit},0},$$

whose injectivity follows from [F].

5.6.   The proof of surjectivity of the map (5.6) is similar to that of [FG4], Proposition 4.13. Namely, we will show that for $\check{\lambda} \in \check{\Lambda}^+$ the map

$$j_{\check{\lambda}+\check{\rho},*} \star \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to j_{\check{\lambda}+\check{\rho},*} \star \mathbb{W}_{\mathrm{crit},0} \simeq \mathbb{W}_{\mathrm{crit},0} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}}{\otimes} \mathcal{L}_{\mathfrak{H},0}^{\check{\lambda}+\check{\rho}}$$

is surjective.

The above map can be obtained by convolution with $j_{\check{\lambda}\cdot w_0,*}$ from the map

$$(5.7) \qquad\qquad j_{w_0\cdot\check{\rho},*} \star \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to j_{w_0\cdot\check{\rho},*} \star \mathbb{W}_{\mathrm{crit},0}.$$

Since the functor $j_{\check{\lambda}\cdot w_0,*}$ is right-exact and sends partially integrable representations to $D^{<0}(\widehat{\mathfrak{g}}_{\mathrm{crit}}-\mathrm{mod}_{\mathrm{reg}})^I$, it is sufficient to show that the cokernel of the map (5.7) is partially integrable. To establish the latter fact, by [FG2], Theorem 18.2.1, it is sufficient to show that the map

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}^-((t)), t\mathfrak{n}[[t]], (j_{w_0\cdot\check{\rho},*}\star\mathbb{V}_{\mathrm{crit}})\otimes\Psi_{-\check{\rho}}\big) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to H^{\frac{\infty}{2}}\big(\mathfrak{n}^-((t)), t\mathfrak{n}[[t]], (j_{w_0\cdot\check{\rho},*}\star\mathbb{W}_{\mathrm{crit},0})\otimes\Psi_{-\check{\rho}}\big)$$

is a surjection. However, by [FG2], 18.1.1, the above map of semi-infinite cohomologies identifies with

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathrm{crit}} \otimes \Psi_0\big) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}_{\mathrm{crit},0} \otimes \Psi_0\big).$$

According to [FB], Theorem 15.1.9, and [FG2], Theorem 18.2.4, the latter map becomes the identity isomorphism under the natural identification of both sides with $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$. This completes the proof of Theorem 4.11.

5.7.   We will now discuss yet another way of obtaining the module $\mathbb{W}_{\mathrm{crit},0}$. Consider the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules

$$(5.8) \qquad\qquad j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}_{-\rho}^{\check{\lambda}}$$

for $\check{\lambda} \in \check{\Lambda}^+$.

We claim that whenever $\check{\lambda}_1 - \check{\lambda}_2 = \check{\mu} \in \check{\Lambda}^+$ there exists a natural map

$$j_{\check{\lambda}_1,*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}_{-\rho}^{\check{\lambda}_1} \to j_{\check{\lambda}_2,*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}_{-\rho}^{\check{\lambda}_2}.$$

Indeed, we have a map of D-modules

$$(5.9) \qquad\qquad j_{\check{\lambda}_1,*} \star \mathcal{F}_{V^{\check{\mu}}} \to j_{\check{\lambda}_1,*} \star j_{\check{\mu},\mathrm{Gr}_G,*} \simeq j_{\check{\lambda}_2,\mathrm{Gr},*},$$

and we compose it with the map

$$\mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}_{\rho}^{\check{\mu}} \overset{\mathrm{id}\,\otimes\kappa^{\check{\mu}}}{\longrightarrow} \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}_{\mathfrak{Z}}^{\check{\mu}} \simeq \mathcal{F}_{V^{\check{\mu}}} \star \mathbb{V}_{\mathrm{crit}}.$$

From the fact that the maps $\kappa^{\check{\mu}}$ satisfy the Plücker relations, it follows that the maps (5.8) form a directed system.

Theorem 5.8. *There exists a canonical isomorphism*

$$\varinjlim j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho} \to \mathbb{W}_{\mathrm{crit},0}.$$

The rest of this section will be devoted to the proof of this theorem. First, we construct a map from the LHS to the RHS. By definition, this amounts to a compatible system of maps

$$j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho} \to \mathbb{W}_{\mathrm{crit},0}.$$

These maps are given by

$$j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho} \to j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho} \to \mathbb{W}_{\mathrm{crit},0},$$

where the first arrow comes by convolution from the canonical map $\phi : \mathbb{V}_{\mathrm{crit}} \to \mathbb{W}_{\mathrm{crit},0}$, and the second map is the isomorphism of Proposition 2.4.

The fact that these maps are compatible for different $\check{\lambda}$ follows from Proposition 2.8.

To construct the map from the RHS to the LHS in Theorem 5.8 we will use Lemma 5.3. For that we need to endow the LHS with an action of the algebra $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{1,\mathfrak{z}})$. Note that this algebra is canonically isomorphic to the direct limit

$$\varinjlim \mathcal{V}^{\check{\lambda}}_{\mathfrak{z}} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho},$$

where the transition maps are defined whenever $\check{\lambda}_2 - \check{\lambda}_1 = \check{\mu} \in \check{\Lambda}^+$ and are equal to

$$\mathcal{V}^{\check{\lambda}_1}_{\mathfrak{z}} \otimes \mathfrak{l}^{\check{\lambda}_1}_{-\rho} \to \mathcal{V}^{\check{\lambda}_1}_{\mathfrak{z}} \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{V}^{\check{\mu}}_{\mathfrak{z}} \otimes \mathfrak{l}^{\check{\mu}}_{-\rho} \otimes \mathfrak{l}^{\check{\lambda}_1}_{-\rho} \to \mathcal{V}^{\check{\lambda}_2}_{\mathfrak{z}} \otimes \mathfrak{l}^{\check{\lambda}_2}_{-\rho},$$

where the first arrow corresponds to the map $\kappa^{\check{\mu}} : \mathfrak{l}^{\check{\mu}}_{\rho} \to \mathcal{V}^{\check{\mu}}_{\mathfrak{z}}$.

We define the desired action of $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{1,\mathfrak{z}})$ on the direct limit appearing in Theorem 5.8 by means of

$$\mathcal{V}^{\check{\mu}}_{\mathfrak{z}} \underset{\mathfrak{z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \left(j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}}_{-\rho} \right) \otimes \mathfrak{l}^{\check{\mu}}_{-\rho} \simeq j_{\check{\lambda},*} \star \mathcal{F}_{V^{\check{\mu}}} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}+\check{\mu}}_{-\rho} \to j_{\check{\lambda}+\check{\mu},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{\check{\lambda}+\check{\mu}}_{-\rho}.$$

The data of morphisms (5.4) for the RHS of Theorem 5.8 is evident from the definitions. It is straightforward to check that the diagram appearing in Lemma 5.3 is commutative, thereby giving rise to a map from the RHS to the LHS in Theorem 5.8. Moreover, it is easy to check that the two maps constructed above are mutually inverse.

6. Wakimoto modules attached to other Schubert cells

In this section we will prove Theorem 4.10. We will derive it from Theorem 4.11 using the technique of chiral modules over chiral algebras.

6.1. **Plan of the proof.** We need to establish an isomorphism

$$(6.1) \qquad\qquad \Gamma^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathcal{W}^{\mathfrak{z}}_w) \simeq \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \mathfrak{l}^{w(\check{\rho})-\check{\rho}}_{w(\rho)}$$

for all $w \in W$. To construct a map in one direction (from left to right) we will use the universal property of $\mathcal{W}^{\mathfrak{z}}_w$, described in Lemma 5.3.

First, we need to make the algebra $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathfrak{z}})$ act on $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}}$. This results from the isomorphism (4.13).

Next, we need to establish that the isomorphism (5.4) holds for $\mathcal{M} = \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}}$. On the one hand, by Proposition 2.4,

$$j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho} \simeq \mathbb{W}_{\mathrm{crit},w(\rho)-\rho} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}{\otimes} \mathcal{L}_{\mathfrak{H},w(\rho)-\rho}^{\check{\lambda}},$$

where

(6.2)
$$\mathcal{L}_{\mathfrak{H},w(\rho)-\rho}^{\check{\lambda}} \simeq \mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \otimes \mathfrak{l}_{w(\rho)}^{\check{\lambda}}.$$

This implies that

$$j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \simeq \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})}{\otimes} \mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}},$$

where we denote

$$\mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}} := \mathcal{L}_{\mathfrak{H},w(\rho)-\rho}^{\check{\lambda}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}.$$

By Sect. 5.2, we need to show that the isomorphism $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}} \simeq \mathrm{Spec}\left(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\right)$

of (4.13) lifts to an isomorphism of $\check{H}$-torsors

(6.3)
$$\{\check{\lambda} \mapsto \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\lambda}}\} \overset{\{\gamma^{\check{\lambda}}\}}{\Longleftrightarrow} \{\check{\lambda} \mapsto \mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}}\}.$$

By (6.2), the latter amounts to an isomorphism of $\check{H}$-torsors

(6.4)
$$\{\check{\lambda} \mapsto \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\lambda}}\} \text{ and } \{\check{\lambda} \mapsto \mathcal{O}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}} \otimes \mathfrak{l}_{w(\rho)}^{\check{\lambda}}\}.$$

6.2. **An identification of $\check{H}$-torsors.** We fill first show that there exists *some* isomorphism as in (6.4) that respects the $\mathrm{Aut}(\mathcal{D})$-actions.

Recall the D-scheme

(6.5)
$$\mathrm{Conn}_{\check{H}}(\omega_X^{\rho})^{\mathrm{RS},\rho-w(\rho)} \underset{\mathrm{Op}_{\check{\mathfrak{g}},X}^{\mathrm{nilp}}}{\times} \mathrm{Op}_{\check{\mathfrak{g}},X}$$

(see Sect. 4.9). Its restriction to $X - x$ is isomorphic to $\mathrm{Conn}_{\check{H}}(\omega_X^{\rho})$, and its fiber over x is isomorphic to $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}$, once we identify $\mathcal{D}_x$ with $\mathcal{D}$.

For $\check{\lambda} \in \check{\Lambda}$, let us denote by $\mathcal{L}_{\mathrm{MOp},w,X}^{\check{\lambda}}$ the line bundle on (6.5) equal to the pull-back from X of the line bundle $\omega_X^{\langle \rho,\check{\lambda}\rangle}\big(\langle \rho - w(\rho),\check{\lambda}\rangle \cdot x\big)$.

We claim that this line bundle identifies with the pull-back of the line bundle $\mathcal{L}_{\mathrm{MOp}_{\check{\mathfrak{g}}},X}^{\check{\lambda}}$ under the map (4.11). This follows from the fact that the corresponding isomorphism holds tautologically over $X - x$, and that both line bundles have connections that are regular at x.

Restricting the above isomorphism of line bundles to $x \in X$, we obtain that $\mathcal{L}_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}}^{\check{\lambda}}$ identifies with $\mathcal{O}_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}} \otimes \omega_x^{\langle w(\rho),\check{\lambda}\rangle}$. The latter isomorphism respects the $\mathrm{Aut}(\mathcal{D})$-actions, where $\mathrm{Aut}(\mathcal{D})$ acts on $\omega_x^{\langle w(\rho),\check{\lambda}\rangle}$ via the isomorphism $\mathrm{Aut}(\mathcal{D}) \simeq \mathrm{Aut}(\mathcal{D}_x)$, corresponding to the above choice of an isomorphism $\mathcal{D} \simeq \mathcal{D}_x$. Finally, let us observe that the character of $\mathrm{Aut}(\mathcal{D})$ on $\omega_x^{\langle w(\rho),\check{\lambda}\rangle}$ equals that of $\mathfrak{l}_{w(\rho)}^{\check{\lambda}}$.

Let us denote by $\mathfrak{H}_{\mathrm{crit},w,X}$ the D-algebra

$$\mathfrak{H}_{\mathrm{crit},w,X} := \mathfrak{H}_{\mathrm{crit},X}^{\mathrm{RS},w_0(w(\rho)-\rho)} \underset{\mathfrak{Z}_{\mathfrak{g},X}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g},X}.$$

Via $\mathrm{map}_{\mathrm{geom}}^{M}$ it identifies with the algebra of regular functions on the D-scheme (6.5). For $\check{\lambda} \in \check{\Lambda}$, let $\mathcal{L}_{\mathfrak{H},w,X}^{\check{\lambda}}$ denote the corresponding line bundle with a *regular* connection along X over it.

The restriction of $\mathcal{L}_{\mathfrak{H},w,X}^{\check{\lambda}}$ to $X - x$ identifies with $\mathcal{L}_{\mathfrak{H},X}^{\check{\lambda}}$. The restriction to the fiber over x, identifies, as a line bundle over $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$ via (4.13), with $\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\lambda}}$. By the above discussion, we obtain an identification

$$\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\lambda}} \simeq \mathcal{O}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}} \otimes \omega_x^{\langle w(\rho),\check{\lambda}\rangle},$$

as required in (6.4).

For future reference note that for $\check{\lambda} \in \check{\Lambda}^+$ we have a canonical map

$$\kappa_{\mathfrak{H}}^{-,\check{\lambda},w} : \mathcal{V}_{\mathfrak{Z},X} \underset{\mathfrak{Z}_{\mathfrak{g}},X}{\otimes} \mathfrak{H}_{\mathrm{crit},w,X} \to \mathcal{L}_{\mathfrak{H},w,X}^{\check{\lambda}}.$$

Its restriction to $X - x$ equals the map $\kappa_{\mathfrak{H}}^{-,\check{\lambda}}$ of (2.9), and its restriction to x coincides with the canonical map (5.2).

6.3.   According to Sect. 5.2, the next step in the definition of the map in (6.1) is construction of a map

$$(6.6) \qquad \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \to \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}.$$

Here $\omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$ is the 1-dimensional $\mathrm{Aut}(\mathcal{D})$-module as in Sect. 6.2. It is non-canonically isomorphic to $\mathfrak{l}_{w(\rho)}^{w(\check{\rho})-\check{\rho}}$. Eventually, we will fix an isomorphism (6.3), thereby fixing a choice for the above isomorphism of lines as well.

Finding a morphism as in (6.6) is equivalent to exhibiting a map

$$(6.7) \qquad \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \to \mathrm{Av}_{G[[t]]/I}\big(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}}\big) \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$$

(see [FG2], Sect. 20.2, for the definition of the averaging functor $\mathrm{Av}_{G[[t]]/I}$).

By the definition of $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}}$ and using the fact that the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}$ is flat over $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\check{\rho})}$, the expression in the RHS of (6.7) can be rewritten as

$$\mathrm{Av}_{G[[t]]/I}\big(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}\big) \overset{L}{\underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\check{\rho})}}{\otimes}} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}.$$

By construction of Wakimoto modules,

$$(6.8) \qquad \mathrm{Av}_{G[[t]]/I}(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}) \simeq H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)),\mathfrak{n}[[t]],\mathfrak{D}^{ch}(G)_{\mathrm{crit},x}\big) \overset{L}{\underset{\mathfrak{h}[[t]];\mathfrak{h}}{\otimes}} \mathbb{C}^{w_0(w(\rho)-\rho)},$$

where $\mathbb{C}^{w_0(w(\rho)-\rho)}$ is the corresponding character of $\mathfrak{h}$.

Consider the component of the expression appearing on the RHS of the above formula that has degree zero with respect to $\mathbb{G}_m \subset \mathrm{Aut}(\mathcal{D})$. It identifies with the $w_0(w(\rho)-\rho)$-weight space in the Lie algebra cohomology $H^\bullet(\mathfrak{n},\mathrm{Fun}(G))$.

By the Bott-Borel Weil theorem, the latter is concentrated in the cohomological degree $\ell(w)$, and is canonically isomorphic to $\mathbb{C}$.

The corresponding vector in the $\ell(w)$-th cohomology of $\mathrm{Av}_{G[[t]]/I}(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho})$ is easily seen to be $\mathfrak{g}[[t]]$-invariant, so we obtain a map

$$\mathbb{V}_{\mathrm{crit}} \to h^{\ell(w)}\big(\mathrm{Av}_{G[[t]]/I}(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho})\big),$$

and, hence, a map

$$(6.9) \quad \left(\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \right) \overset{L}{\underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\check{\rho})}}{\otimes}} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}[-\ell(w)] \to$$

$$\to \mathrm{Av}_{G[[t]]/I}\left(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho} \right) \overset{L}{\underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\check{\rho})}}{\otimes}} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}.$$

We claim that the 0th cohomology of the LHS of the expression in (6.9) is canonically isomorphic to $\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$, which would produce the desired (non-zero!) map in (6.7).

To establish this isomorphism, it would be enough to show that

$$(6.10) \qquad \mathrm{Tor}_{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}^{\ell(w)}\left(\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}), \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \right) \simeq \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \otimes \omega_x^{\langle w(\rho)-\rho, \check{\rho} \rangle}.$$

We can identify the LHS of (6.10) with $\Lambda^{\ell(w)}\left(N^*_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}} / \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)})} \right)$, and the RHS with $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z},w}^{\check{G}}}^{\check{\rho}-w(\check{\rho})}$, using Sect. 6.2. Hence, (6.10) is equivalent to an isomorphism of line bundles

$$\Lambda^{\ell(w)}\left(N^*_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}} / \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)})} \right) \simeq \mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z},w}^{\check{G}}}^{\check{\rho}-w(\check{\rho})}$$

over $\mathrm{Fl}_{\mathfrak{Z},w}^{\check{G}}$. Using the diagram

$$\begin{array}{ccc}
\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}} & \longrightarrow & \mathrm{Spec}\left(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)} \right) \\
\sim \downarrow & & \sim \downarrow \\
\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}} & \longrightarrow & \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{nilp}},
\end{array}$$

we can translate the existence of the above isomorphism to that between the line bundles

$$\Lambda^{\ell(w)}\left(N^*_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}} / \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{nilp}}} \right) \simeq \mathcal{L}_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}}^{\check{\rho}-w(\check{\rho})}$$

over $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}$. The latter isomorphism follows from [FG2], Corollary 3.6.3.

We will need the following property of the map (6.6):

Proposition 6.4.
(1) *The map*

$$H^{\frac{\infty}{2}}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathrm{crit}} \otimes \Psi_0 \right) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \to$$

$$H^{\frac{\infty}{2}}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \Psi_0 \right) \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle},$$

induced by (6.6), *is an isomorphism.*
(2) *The map*

$$H^{\frac{\infty}{2}}\left(\mathfrak{n}^-((t)), t\mathfrak{n}^-[[t]], j_{w_0 \cdot \check{\rho},*} \star \mathbb{V}_{\mathrm{crit}} \otimes \Psi_{-\check{\rho}} \right) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \to$$

$$H^{\frac{\infty}{2}}\left(\mathfrak{n}^-((t)), t\mathfrak{n}^-[[t]], j_{w_0 \cdot \check{\rho},*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \Psi_{-\check{\rho}} \right) \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}$$

is also an isomorphism.

The proof of this proposition will be given in Sect. 6.11.

6.5. By Lemma 5.3, in order to complete the construction of the map in (6.1) we need to choose an isomorphism in (6.3) so that the following diagram becomes commutative for every $\check{\lambda} \in \check{\Lambda}^+$:

(6.11)

$$
\begin{array}{ccc}
\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{V}^{\check{\lambda}}_{\mathfrak{Z}} & \longrightarrow & \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{V}^{\check{\lambda}}_{\mathfrak{Z}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle} \\
{\scriptstyle \sim}\downarrow & & {\scriptstyle \kappa^{-,\check{\lambda},w}}\downarrow \\
\mathcal{F}_{V^{\check{\lambda}}} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}} & \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}})}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle} \\
\downarrow & & {\scriptstyle \sim}\downarrow {\scriptstyle \gamma^{\check{\lambda}}} \\
j_{\check{\lambda},\mathrm{Gr}_G,*} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}} & \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}})}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathfrak{H},w,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle} \\
{\scriptstyle \sim}\downarrow & & {\scriptstyle \sim}\downarrow \\
j_{\check{\lambda},*} \star \mathbb{V}_{\mathrm{crit}} & \longrightarrow & j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle},
\end{array}
$$

where $\kappa^{-,\check{\lambda},w}$ is as in (5.2).

To construct the isomorphism (6.3), we will realize $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$ as a fiber at $x \in X$ of a certain chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module.

6.6. We will use the following general construction. Let $\mathcal{A}$ be a chiral algebra, let $\mathcal{M}$ be a torsion-free chiral $\mathcal{A}$-module on $X - x$, and let $\mathcal{N}_1, \mathcal{N}_2$ be two chiral $\mathcal{A}$-modules, supported at x. Let

(6.12) $$\jmath_*(\mathcal{M}) \otimes \mathcal{N}_1 \to \imath_!(\mathcal{N}_2)$$

be a chiral pairing (see [CHA] and [FG3], Sect. 2.1). Here $\jmath$ and $\imath$ denote the embeddings $X - x \hookrightarrow X$ and $x \hookrightarrow X$, respectively.

Let $\mathbf{v} \in \mathcal{N}_1$ be a vector, which is annihilated by the Lie-$*$ action of $\mathcal{A}$, and such that the resulting map $\jmath_*(\mathcal{M}) \to \imath_!(\mathcal{N}_2)$ is surjective. Then

$$\mathcal{M}' := \ker\big(\jmath_*(\mathcal{M}) \to \imath_!(\mathcal{N}_2)\big)$$

is a chiral $\mathcal{A}$-module, whose fiber at x is $\mathcal{N}_2$.

6.7. We apply the above construction in the following situation. We let $\mathcal{M}$ be $\mathbb{W}_{\mathrm{crit},X}$. Recall (see [F] and [FG2], Sect. 10.3), that by the construction of Wakimoto modules, $\mathbb{W}_{\mathrm{crit},X}$ is in fact a chiral algebra that contains $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$ as a chiral subalgebra.

We let $\mathcal{N}_1 = \mathcal{N}_2 := \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$, which is naturally a chiral $\mathbb{W}_{\mathrm{crit},X}$-module, supported at $x \in X$. Finally, we let $\mathbf{v}$ to be the image of the vacuum vector in $\mathbb{V}_{\mathrm{crit}}$ under the map (6.6).

Lemma 6.8. *The resulting map*

$$\jmath_*(\mathbb{W}_{\mathrm{crit},W}) \to \imath_!\left(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}\right)$$

is surjective.

Proof. From the construction of Wakimoto modules we obtain that $\mathbb{W}_{\mathrm{crit},X}$-submodules of $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$ are in bijection with $\mathfrak{H}_{\mathrm{crit},X}$-submodules of $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}) \otimes \omega_x^{\langle \rho-w(\rho),\check{\rho}\rangle}$, and the correspondence is given by applying the functor $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$.

Hence, the assertion of the lemma follows from Proposition 6.4(1).

$\square$

Let us denote the resulting chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-module by $\mathbb{W}_{\mathrm{crit},w,X}$. By construction, it comes equipped with a map of chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-modules $\mathbb{V}_{\mathrm{crit},X} \to \mathbb{W}_{\mathrm{crit},w,X}$.

By construction, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{z}_{\mathfrak{g},X} & \longrightarrow & \mathbb{V}_{\mathrm{crit},X} \\
\downarrow & & \downarrow \\
\mathfrak{H}_{\mathrm{crit},w,X} & \longrightarrow & \mathbb{W}_{\mathrm{crit},w,X},
\end{array}$$

whose restriction to $X - x$ is

$$\begin{array}{ccc}
\mathfrak{z}_{\mathfrak{g},X-x} & \longrightarrow & \mathbb{V}_{\mathrm{crit},X-x} \\
\varphi \downarrow & & \phi \downarrow \\
\mathfrak{H}_{\mathrm{crit},X-x} & \longrightarrow & \mathbb{W}_{\mathrm{crit},X-x},
\end{array}$$

and whose fiber at x is the diagram

$$\begin{array}{ccc}
\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}} & \longrightarrow & \mathbb{V}_{\mathrm{crit}} \\
\downarrow & & \downarrow {\scriptstyle (6.6)} \\
\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}) & \longrightarrow & \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}.
\end{array}$$

6.9.   For $\check{\lambda} \in \check{\Lambda}^+$ consider the chiral module $j_{\check{\lambda},*,X} \star \mathbb{W}_{\mathrm{crit},w,X}$. We claim that this convolution is concentrated in the cohomological degree 0.

Indeed, *a priori*, it is concentrated in non-positive cohomolgical degrees, since the functor $j_{\check{\lambda},*,X} \star ?$ is right-exact. Now, it does not have strictly negative cohomologies, because this is true for both chiral $\mathcal{A}_{\mathfrak{g},\mathrm{crit},X}$-modules $\jmath_*(\mathbb{W}_{\mathrm{crit},X})$ and $\imath_!\left(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}\right)$.

Recall the $\check{H}$-torsor $\{\check{\lambda} \mapsto \mathcal{L}_{\mathfrak{H},w,X}^{\check{\lambda}}\}$ on $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit},w,X})$ (see Sect. 6.2).

Proposition 6.10.
(1) *The map* $\mathcal{L}_{\mathfrak{H},X-x}^{\check{\lambda}} \to j_{\check{\lambda},X-x,*} \star \mathbb{W}_{\mathrm{crit},X-x}$, *equal to the composition*

$$\mathcal{L}_{\mathfrak{H},X-x}^{\check{\lambda}} \hookrightarrow \mathcal{L}_{\mathfrak{H},X-x}^{\check{\lambda}} \underset{\mathfrak{H}_{\mathrm{crit},X-x}}{\otimes} \mathbb{W}_{\mathrm{crit},X-x} \simeq j_{\check{\lambda},X-x,*} \star \mathbb{W}_{\mathrm{crit},X-x},$$

extends to a map $\mathcal{L}_{\mathfrak{H},w,X}^{\check{\lambda}} \to j_{\check{\lambda},*,X} \star \mathbb{W}_{\mathrm{crit},w,X}$.
(2) *The resulting map on the level of fibers at x*

$$\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}}^{\check{\lambda}} \to j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle} \simeq$$

$$\simeq \mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}})}{\otimes} \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}$$

comes from an isomorphism *onto* $\mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}}$, *followed by the map*

$$\mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}} \hookrightarrow \mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}} \underset{\mathrm{Fun}(\mathrm{Fl}_{\mathfrak{z},w}^{\check{G}})}{\otimes} \left(\mathrm{Fun}(\mathrm{Fl}_{\mathfrak{z},w}^{\check{G}}) \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathbb{V}_{\mathrm{crit}} \right) \xrightarrow{(6.6)}$$

$$\to \mathcal{L}_{\mathfrak{H},w,\mathrm{reg}}^{\check{\lambda}} \underset{\mathrm{Fun}(\mathrm{Fl}_{\mathfrak{z},w}^{\check{G}})}{\otimes} \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}.$$

Let us assume this proposition and finish the construction of the map in (6.1). First, note that the second assertion of the proposition defines the sought-after identification $\gamma^{\check\lambda}$ of (6.3).

Next, observe that the first assertion of the proposition implies that there exists a diagram:

$$(6.13)$$

$$
\begin{array}{ccc}
\mathcal{F}_{V^{\check\lambda},X} \star \mathbb{V}_{\mathrm{crit},X} & \xleftarrow{\ \sim\ } & \mathbb{V}_{\mathrm{crit},X} \underset{\mathfrak{z}_{\mathfrak{g}},X}{\otimes} \mathcal{V}_{\mathfrak{Z},X} \\
\downarrow & & \downarrow \\
j_{\check\lambda,*,X} \star \mathbb{V}_{\mathrm{crit},X} & & \mathbb{V}_{\mathrm{crit},X} \underset{\mathfrak{z}_{\mathfrak{g}},X}{\otimes} \mathcal{V}_{\mathfrak{Z},X} \underset{\mathfrak{z}_{\mathfrak{g}},X}{\otimes} \mathfrak{H}_{\mathrm{crit},w,X} \\
\downarrow & & \ \ \downarrow{\scriptstyle \kappa_{\mathfrak{H}}^{-,\lambda,w}} \\
j_{\check\lambda,*,X} \star \mathbb{W}_{\mathrm{crit},w,X} & \longleftarrow & \mathbb{V}_{\mathrm{crit},X} \underset{\mathfrak{z}_{\mathfrak{g}},X}{\otimes} \mathcal{L}^{\check\lambda}_{\mathfrak{H},w,X},
\end{array}
$$

which is commutative, since it extends the corresponding commutative diagram over $X - x$.

Let us consider the fiber of the above diagram over x. The two resulting maps

$$
\mathbb{V}_{\mathrm{crit},X} \underset{\mathfrak{z}_{\mathfrak{g}},X}{\otimes} \mathcal{V}_{\mathfrak{Z},X} \rightrightarrows \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{\mathrm{Fun}(\mathrm{Fl}^{\check G}_{w,\mathfrak{Z}})}{\otimes} \mathcal{L}^{\check\lambda}_{\mathfrak{H},w,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho),\check\rho\rangle}
$$

are equal to the two circuits in the diagram (6.11). Thus, we obtain a well-defined map (6.1).

The proof that this map is an isomorphism uses the same argument as the one used in the proof of Theorem 4.11 in Sect. 5.5:

The surjectivity assertion follows from Proposition 6.4(2) as in *loc. cit.* To prove the injectivity, it is enough to show that the map (6.6) itself is injective.

However, according to [FG1], the functor

$$
\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}^{G[[t]]}, \qquad \mathcal{T} \mapsto \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{T}
$$

is an equivalence of categories. Therefore for any $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-module $\mathcal{T}$, the submodules of $\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{T}$ are in bijection with the $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-submodules of $\mathcal{T}$ itself, and any such submodule is determined by its image in

$$
\mathcal{T} \simeq H^{\frac{\infty}{2}}\Big(\mathfrak{n}((t)), \mathfrak{n}[[t]], (\mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{T}) \otimes \Psi_0 \Big).
$$

Therefore, the injectivity of (6.6) follows from Proposition 6.4(1).

This completes the proof of Theorem 4.10 modulo Proposition 6.4 and Proposition 6.10. The remainder of this section is devoted to the proof these two propositions.

6.11. Proof of Proposition 6.4.

First, let us notice that assertions (1) and (2) of the proposition are equivalent by [FG2], Proposition 18.1.1.

Secondly, both the LHS and the RHS, appearing in Proposition 6.4(2), are isomorphic to $\mathrm{Fun}(\mathrm{Fl}^{\check G}_{w,\mathfrak{Z}}) \otimes \omega_x^{\langle\rho,\check\rho\rangle}$, as graded $\mathrm{Fun}(\mathrm{Fl}^{\check G}_{w,\mathfrak{Z}})$-modules. Since the degree 0 component of $\mathrm{Fun}(\mathrm{Fl}^{\check G}_{w,\mathfrak{Z}})$ consists of scalars, it sufficient to show that the map in Proposition 6.4(1) is non-zero.

Let us assume the contrary. Then, by [FG2], Theorem 18.2.1, the map

$$
j_{w_0\cdot\check\rho,*} \star \mathbb{V}_{\mathrm{crit}} \to j_{w_0\cdot\check\rho,*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho),\check\rho\rangle}
$$

has a partially integrable image. Let us denote it by $\mathcal{M}$, and consider the composition

$$
j_{\check\lambda\cdot w_0,*} \star j_{w_0\cdot\check\rho,*} \star \mathbb{V}_{\mathrm{crit}} \to j_{\check\lambda\cdot w_0,*} \star \mathcal{M} \to j_{\check\lambda\cdot w_0,*} \star j_{w_0\cdot\check\rho,*} \star \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega_x^{\langle \rho - w(\rho),\check\rho\rangle}
$$

for $\check{\lambda} \in \check{\Lambda}^+$. On the one hand, $j_{\check{\lambda} \cdot w_0, *} \star j_{w_0 \cdot \check{\rho}, *} \star \mathbb{V}_{\text{crit}} \simeq j_{\check{\lambda}, *} \star \mathbb{V}_{\text{crit}}$ and

$$j_{\check{\lambda} \cdot w_0, *} \star j_{w_0 \cdot \check{\rho}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle} \simeq j_{\check{\lambda}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle},$$

and the above composed map is non-zero, since the functor $j_{\check{\lambda}, *} \star ?$ is invertible on the derived category.

On the other hand, since $\mathcal{M}$ is partially integrable, the convolution $j_{\check{\lambda} \cdot w_0, *} \star \mathcal{M}$ belongs to $D^{<0}(\widehat{\mathfrak{g}}_{\text{crit}} - \text{mod})$, and its map to $j_{\check{\lambda}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle} \in \widehat{\mathfrak{g}}_{\text{crit}} - \text{mod}$ is necessarily 0.

6.12. **Proof of Proposition 6.10.** Consider the D-submodule of $j_{\check{\lambda}, *, X} \star \mathbb{W}_{\text{crit}, w, X}$ consisting of sections which are annihilated by the Lie-* action of $\mathcal{A}_{\mathfrak{g}, \text{crit}, X}$. This is a torsion-free chiral module over $\mathfrak{H}_{\text{crit}, w, X}$. The restriction of this D-module to $X - x$ identifies canonically with $\mathcal{L}_{\mathfrak{H}, X}^{\check{\lambda}}$. The claim about the extension follows now from [FG3], Proposition 3.4.

To prove the second assertion of the proposition, we claim that it is enough to show that the resulting map

$$(6.14) \qquad \mathcal{L}_{\text{Fl}_{w,3}^{\check{G}}}^{\check{\lambda}} \to j_{\check{\lambda}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}$$

is non-zero.

Indeed, dividing by the line $\omega_x^{\langle w(\rho), \check{\lambda} \rangle}$, we obtain that the map in Proposition 6.10(2) corresponds to a non-zero $G[[t]]$-invariant vector in $\mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}$, which has degree 0 with respect to $\mathbb{G}_m \subset \text{Aut}(\mathcal{D})$. Hence it must coincide, up to a scalar, with the spherical vector used in the construction of the map (6.6).

Suppose that the map (6.14) was zero. We would obtain that in the commutative diagram (6.13) the composed map

$$\mathcal{F}_{V^{\check{\lambda}}, X} \star \mathbb{V}_{\text{crit}, X} \to j_{\check{\lambda}, *, X} \star \mathbb{W}_{\text{crit}, w, X}$$

is such that its fiber at x is zero.

In other words, we obtain that the map

$$\mathcal{F}_{V^{\check{\lambda}}} \star \mathbb{V}_{\text{crit}} \to j_{\check{\lambda}, *} \star \mathbb{V}_{\text{crit}} \to j_{\check{\lambda}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle},$$

obtained from (6.6), vanishes.

Consider the map

$$(6.15) \qquad j_{w_0 \cdot \check{\rho}, *} \star j_{\check{\lambda}, *} \star \mathbb{V}_{\text{crit}} \to j_{w_0 \cdot \check{\rho}, *} \star j_{\check{\lambda}, *} \star \mathbb{W}_{\text{crit}, w(\rho) - \rho, \text{reg}} \otimes \omega_x^{\langle \rho - w(\rho), \check{\rho} \rangle}.$$

We obtain that its composition with

$$\text{IC}_{w_0 \cdot (\check{\rho} + \check{\lambda}), \text{Gr}_G} \star \mathbb{V}_{\text{crit}} \to j_{w_0 \cdot (\check{\rho} + \check{\lambda}), \text{Gr}_G, *} \star \mathbb{V}_{\text{crit}} \simeq j_{w_0 \cdot \check{\rho}, *} \star j_{\check{\lambda}, *} \star \mathbb{V}_{\text{crit}}$$

vanishes.

However, by [ABBGM], Sect. 2.3, the cokernel $j_{w_0 \cdot (\check{\rho} + \check{\lambda}), \text{Gr}_G, *} / \text{IC}_{w_0 \cdot (\check{\rho} + \check{\lambda}), \text{Gr}_G}$ is partially integrable. Hence, the image of the map in (6.15) is partially integrable. But this leads to a contradiction as in the proof of Proposition 6.4 given above.

7. The general case

7.1.   In this section we will carry out one more step in the proof Theorem 4.8. Namely, we will prove that an isomorphism

$$(7.1) \qquad {}_w\mathbb{W}(\mathfrak{T}) \simeq \mathsf{G}\Big(\mathcal{L}_w^{\mathrm{twist}} \underset{\mathcal{O}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}}}}{\otimes} \mathfrak{T}\Big),$$

holds functorially in $\mathfrak{T} \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}})$ for *some* bi-module $\mathcal{L}_w^{\mathrm{twist}}$ over the topological commutative algebra $\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}})$. Moreover, we will show that as a *right* module, $\mathcal{L}_w^{\mathrm{twist}}$ is a line bundle over $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}}$.

As an immediate corollary of the above isomorphism, combined with Sect. 4.3, we obtain the following:

Corollary 7.2. *For every* $\mathfrak{T} \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}})$ *the Wakimoto module* ${}_w\mathbb{W}(\mathfrak{T})$ *is I-equivariant.*

The main step in the proof of (7.1) will be the following:

Proposition 7.3. *Let* $\mathfrak{T}^1$ *and* $\mathfrak{T}^2$ *be two quasi-coherent sheaves on* $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$. *Then the functor* $\mathsf{E}^{\mathfrak{z}} : D^+\big(\mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{z}}^{\check{G}})\big) \to D^+\big(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{z}}} -\mathrm{mod}^{I^0}\big)$ *induces a bijection*

$$\mathrm{Ext}_{\mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{z}}^{\check{G}})}^i(\mathfrak{T}^1, \mathfrak{T}^2) \to R^i \mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{z}}} -\mathrm{mod}^{I^0}}\big(\mathsf{E}^{\mathfrak{z}}(\mathfrak{T}^1), \mathsf{E}^{\mathfrak{z}}(\mathfrak{T}^2)\big)$$

for $i = 0, 1$.

Note that the map in the proposition is *not* bijective for $i \geq 2$, for otherwise the functor $\mathsf{E}^{\mathfrak{z}}$ would be an equivalence of categories, which it is not.

Let us show how this proposition implies (7.1).

Step 1. We claim that ${}_w\mathbb{W}(\mathfrak{T})$ is isomorphic to $\mathsf{G}(\mathfrak{T}') := \Gamma^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathrm{Gr}_G, \mathsf{E}^{\mathfrak{z}}(\mathfrak{T}'))$ for *some* object $\mathfrak{T}' \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}})$.

Without loss of generality we can assume that $\mathfrak{T}$ is supported on k-th infinitesimal neighborhood of $\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}$ in $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$. We will argue by induction on k.

If $k = 0$, i.e., when $\mathfrak{T}$ is scheme-theoretically supported on $\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}$, we have:

$${}_w\mathbb{W}(\mathfrak{T}) \simeq \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}})}{\otimes} \mathfrak{T},$$

and the assertion follows from Theorem 4.10.

Suppose now that $k > 1$. Then we can write $\mathfrak{T}$ as an extension

$$0 \to \mathfrak{T}^1 \to \mathfrak{T} \to \mathfrak{T}^2 \to 0,$$

with $\mathfrak{T}^1$ supported on on $\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}$ and $\mathfrak{T}^2$ supported on the $k-1$-st infinitesimal neighborhood of $\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}$ in $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$. Then, by the induction hypothesis, there exist objects $\mathfrak{T}'^1, \mathfrak{T}'^2 \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{z}}^{\check{G}})$, such that

$$\mathsf{G}(\mathfrak{T}'^i) \simeq {}_w\mathbb{W}(\mathfrak{T}^i)$$

for $i = 1, 2$.

By the main theorem of [FG4], the extension

$$0 \to {}_w\mathbb{W}(\mathfrak{T}^1) \to {}_w\mathbb{W}(\mathfrak{T}) \to {}_w\mathbb{W}(\mathfrak{T}^2) \to 0$$

comes from some extension

$$0 \to \mathsf{E}^{\mathfrak{z}}(\mathfrak{T}'^1) \to \mathfrak{F} \to \mathsf{E}^{\mathfrak{z}}(\mathfrak{T}'^2) \to 0$$

in $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\check{3}}}\mathrm{-mod}^{I^0}$. However, by Proposition 7.3

$$\mathcal{F} \simeq \mathsf{E}^{\check{3}}(\mathcal{T}')$$

for some $\mathcal{T}' \in \mathrm{QCoh}(\mathrm{Fl}_{\check{3}}^{\check{G}})$, as required. Evidently, $\mathcal{T}'$, being an extension of two quasi-coherent sheaves that belong to $\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})$, itself belongs to this subcategory.

Step 2. Proposition 7.3 for $i = 0$, combined with the equivalence of [FG4], implies that the assignment $\mathcal{T} \mapsto \mathcal{T}'$, constructed above, is a functor $\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}}) \to \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})$; let us denote it by ${}_w\mathsf{Q}$. Since each of the functors ${}_w\mathbb{W}$, $\Gamma^{\mathrm{Hecke}_{\check{3}}}$ and $\mathsf{E}|_{\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})}$ is exact and faithful, we obtain that ${}_w\mathsf{Q}$ is also exact and faithful.

Hence, ${}_w\mathsf{Q}$ has the form

$$\mathcal{T} \mapsto \mathcal{L}_w^{\mathrm{twist}} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})}{\otimes} \mathcal{T},$$

for a certain $\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})$-bimodule $\mathcal{L}_w^{\mathrm{twist}}$. We will show now that as a *right* $\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\check{3}}^{\check{G}})$-module, $\mathcal{L}_w^{\mathrm{twist}}$ is a line bundle. In fact, we will show that it is non-canonically trivial.

This is equivalent to showing that there exists a functorial isomorphism between the vector space underlying ${}_w\mathsf{Q}(\mathcal{T})$ and that of $\mathcal{T}$. We will do this by comparing the semi-infinite cohomologies $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$.

Step 3.

Recall that by [FG2], Sect. 12.4, there exists an isomorphism

$$(7.2) \qquad\qquad H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{W}(\mathcal{T}) \otimes \Psi_0) \simeq \mathcal{T},$$

which is functorial in $\mathcal{T} \in \widehat{\mathfrak{H}}_{\mathrm{crit}}\mathrm{-mod}$, but it is non-canonical in the sense that it depends on the choice of the coordinate t on $\mathcal{D}$.

Thus, to identify ${}_w\mathsf{Q}(\mathcal{T})$ and $\mathcal{T}$ as vector spaces, it suffices to prove the following:

Proposition 7.4. *For any* $\mathcal{T}' \in \mathrm{QCoh}(\mathrm{Fl}_{\check{3}}^{\check{G}})$, *there exists a canonical quasi-isomorphism*

$$H^{\frac{\infty}{2}+\bullet}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathsf{G}(\mathcal{T}') \otimes \Psi_0\right) \simeq R\Gamma(\mathrm{Fl}_{\check{3}}^{\check{G}}, \mathcal{T}').$$

Proof. By the definition of convolution, the LHS of the proposition is given by applying the functor of derived $\check{B}_{\check{3}}$-invariants to the complex

$$(7.3) \qquad H^{\frac{\infty}{2}+\bullet}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma^{\mathrm{Hecke}_{\check{3}}}\left(\mathrm{Gr}_G, \mathrm{act}^*(\mathcal{W}_{w_0}^{\check{3}}) \underset{\mathrm{Fun}(\check{G}_{\check{3}})}{\otimes} \widetilde{\mathcal{T}'}\right) \otimes \Psi_0\right),$$

where $\widetilde{\mathcal{T}'}$ denotes the pull-back of $\mathcal{T}'$ to $\check{G}_{\check{3}}$.

However, by [FG4], Sect. 2.12,

$$H^{\frac{\infty}{2}+\bullet}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma^{\mathrm{Hecke}_{\check{3}}}\left(\mathrm{Gr}_G, \mathrm{act}^*(\mathcal{W}_{w_0^{\check{3}}})\right) \otimes \Psi_0\right) \simeq$$

$$\simeq H^{\frac{\infty}{2}+\bullet}\left(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma^{\mathrm{Hecke}_{\check{3}}}(\mathrm{Gr}_G, \mathcal{W}_{w_0}^{\check{3}}) \otimes \Psi_0\right) \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\check{G}_{\check{3}}) \simeq \mathrm{Fun}(\check{G}_{\check{3}}).$$

Hence, the expression in (7.3) is isomorphic to $\widetilde{\mathcal{T}'}$, as a $\mathrm{Fun}(\check{G}_{\check{3}})$-module, endowed with a $\check{B}_{\check{3}}$-action. Finally, we have:

$$R\,\mathrm{Inv}(\check{B}_{\check{3}}, \widetilde{\mathcal{T}'}) \simeq R\Gamma(\mathrm{Fl}_{\check{3}}^{\check{G}}, \mathcal{T}'),$$

which is what we had to show.

$\square$

7.5. **Proof of Proposition 7.3.**

Step 1. We claim that it is enough to prove a version of the proposition for the categories $(\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}, \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}} \mathrm{-mod}^{I^0})$ replaced by $(\mathrm{Fl}^{\check{G}}, \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0})$, i.e., that for $\mathcal{T}^1, \mathcal{T}^2 \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ the map

$$(7.4) \qquad \mathrm{Ext}^i_{\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})}(\mathcal{T}^1, \mathcal{T}^2) \to R^i \mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0}}\big(\mathsf{E}(\mathcal{T}^1), \mathsf{E}(\mathcal{T}^2)\big)$$

is an isomorphism for $i = 0, 1$.

Indeed, by choosing a trivialization of the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$, we can identify $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$ (resp., $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}} \mathrm{-mod}^{I^0}$) with the category of objects of $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$ (resp., $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0}$), endowed with an action of the algebra $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$, and the functor $\mathsf{E}^{\mathfrak{Z}}$ is obtained from E by extension of scalars.

The fact that (7.4) is an isomorphism for $i = 0, 1$ is a formal corollary of [ABG]. Below we will give an independent proof.

Step 2. We claim that it is enough to show that the maps

$$(7.5) \qquad \mathrm{Ext}^i_{\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})}\big(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}, \mathcal{T}\big) \to R^i \mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0}}\big(\mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}), \mathsf{E}(\mathcal{T})\big)$$

are isomorphisms for $i = 0, 1$, $\check{\lambda} \in \check{\Lambda}^+$, and any $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$.

Without loss of generality we can assume that $\mathcal{T}^1$ is coherent, and let

$$\ldots \to \mathcal{Q}^2 \to \mathcal{Q}^1 \to \mathcal{Q}^0 \to \mathcal{T}^1 \to 0,$$

be a resolution which each $\mathcal{Q}^i$ is isomorphic to a direct sum of line bundles $\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}_i^j}$.

The cohomological dimension of the functor E is *a priori* bounded by the cohomological dimension of the category $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$, which is $\dim(\mathrm{Fl}^{\check{G}})$. Let $\check{\lambda} \in \check{\Lambda}^+$ be such that $\check{\lambda}_i^j + \check{\lambda} \in \check{\Lambda}^+$ for $i \leq \dim(\mathrm{Fl}^{\check{G}}) + 1$. Then by Proposition 3.18, $\mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}_i^j + \check{\lambda}}) \simeq \mathrm{Ind}^{\mathrm{Hecke}}(j_{\check{\lambda}_i^j + \check{\lambda}, \mathrm{Gr}_G, *})$. In particular, this implies that $\mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{T}) \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0}$, i.e., $\mathsf{E}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{T})$ does not have higher cohomologies.

By Proposition 3.19, for $\mathcal{T}^1, \mathcal{T}^2$ as above we have a commutative diagram

$$
\begin{array}{ccc}
R\mathrm{Hom}(\mathcal{T}^1, \mathcal{T}^2) & \xrightarrow{\;\;\mathsf{E}\;\;} & R\mathrm{Hom}\big(\mathsf{E}(\mathcal{T}^1), \mathsf{E}(\mathcal{T}^2)\big) \\
\sim \downarrow & & \downarrow \\
R\mathrm{Hom}(\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{T}^1, \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{T}^2) & \xrightarrow{\;\;\mathsf{E}\;\;} & R\mathrm{Hom}\big(j_{\check{\lambda},*} \star \mathsf{E}(\mathcal{T}^1), j_{\check{\lambda},*} \star \mathsf{E}(\mathcal{T}^2)\big).
\end{array}
$$

Now notice that the functor $\widetilde{j}_{\check{\lambda},*} \underset{I^0}{\star} ?$ is a self-equivalence of $D\big(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}} \mathrm{-mod}^{I^0}\big)$. (Its quasi-inverse is given by the !-convolution with $\widetilde{j}_{-\check{\lambda},*}$.) Hence, the right vertical arrow in the above diagram is an isomorphism as well.

Hence, we can replace the initial sheaves $\mathcal{T}^1$ and $\mathcal{T}^2$ by their twists with respect to $\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}}$. Moreover, we can use the 3-term resolution

$$\mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{Q}_2 \to \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{Q}_1 \to \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{Q}_0 \to \mathcal{L}_{\mathrm{Fl}^{\check{G}}}^{\check{\lambda}} \otimes \mathcal{T}^1$$

to compute both sides of (7.4). This performs the required reduction in Step 2.

Using Proposition 3.19 again, we reduce the assertion further to the case $\check{\lambda} = 0$.

Step 3. Let us recall the general set-up of Sect. 3.5. Let $\mathcal{M}$ (resp., $\mathcal{N}$) be a $\check{G}$-equivariant (resp., $\check{B}$-equivariant) object of $\mathcal{C}$, and let $\mathcal{T}$ be an object of $\mathrm{QCoh}(\mathrm{Fl}^{\check{G}})$. In this case $R\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$

is naturally an object of the derived category of $\check{B}$-modules. Let $\widetilde{R\operatorname{Hom}}(\mathcal{M},\mathcal{N})$ denote the associated complex of $\check{G}$-equivariant quasi-coherent sheaves on $\operatorname{Fl}^{\check{G}}$.

For $\mathcal{T} \in \operatorname{QCoh}(\operatorname{Fl}^{\check{G}})$ we have:

$$(7.6) \qquad R\operatorname{Hom}_{\mathcal{C}}\left(\mathcal{M}, \mathcal{T} \overset{R}{*} \mathcal{N}\right) \simeq R\Gamma\left(\operatorname{Fl}^{\check{G}}, \mathcal{T} \underset{\mathcal{O}_{\operatorname{Fl}^{\check{G}}}}{\otimes} \widetilde{R\operatorname{Hom}}(\mathcal{M},\mathcal{N})\right).$$

Applying this to $\{\mathcal{C},\mathcal{M},\mathcal{N}\})$ being

$$\{\operatorname{QCoh}(\operatorname{Fl}^{\check{G}}), \mathcal{O}_{\operatorname{Fl}^{\check{G}}}, \mathbb{C}_{w_0}\} \text{ and } \{\operatorname{D}(\operatorname{Gr}_G)^{\operatorname{Hecke}}_{\operatorname{crit}}\!-\!\operatorname{mod}^{I^0}, \mathsf{E}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}), \mathsf{E}(\mathbb{C}_{w_0})\},$$

we obtain a commutative diagram

$$
\begin{array}{ccc}
R\operatorname{Hom}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}, \mathcal{T}) & \xrightarrow{\ (7.4)\ } & R\operatorname{Hom}_{\operatorname{D}(\operatorname{Gr}_G)^{\operatorname{Hecke}}_{\operatorname{crit}}\!-\!\operatorname{mod}^{I^0}}\left(\mathsf{E}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}), \mathsf{E}(\mathcal{T})\right) \\[2mm]
\sim\Big\downarrow & & \sim\Big\downarrow \\[2mm]
R\Gamma\left(\operatorname{Fl}^{\check{G}}, \mathcal{T} \underset{\mathcal{O}_{\operatorname{Fl}^{\check{G}}}}{\otimes} \widetilde{R\operatorname{Hom}}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}, \mathbb{C}_{w_0})\right) & \longrightarrow & R\Gamma\left(\operatorname{Fl}^{\check{G}}, \mathcal{T} \underset{\mathcal{O}_{\operatorname{Fl}^{\check{G}}}}{\otimes} \widetilde{R\operatorname{Hom}}(\mathsf{E}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}), \mathsf{E}(\mathbb{C}_{w_0}))\right)
\end{array}
$$

Thus, we obtain that it is enough to show that (7.5) is an isomorphism for $i = 0, 1$, $\check{\lambda} = 0$ and $\mathcal{T} = \mathbb{C}_{w_0}$.

Let $\operatorname{Fl}^{\check{G}}_1 \subset \operatorname{Fl}^{\check{G}}$ be the open Schubert cell, and let us apply the above commutative diagram again with $\mathcal{T} = \mathcal{O}_{\operatorname{Fl}^{\check{G}}_1}$. Since $\operatorname{Fl}^{\check{G}}_1$ is affine, we obtain that the isomorphism of (7.5) for $\mathcal{T} = \mathbb{C}_{w_0}$ will follow once we establish it for $\mathcal{T} = \mathcal{O}_{\operatorname{Fl}^{\check{G}}_1}$.

Step 4. Let us denote $\mathsf{E}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}_1})$ by $\mathcal{W}_1$, and recall that $\mathsf{E}(\mathcal{O}_{\operatorname{Fl}^{\check{G}}}) \simeq \operatorname{Ind}^{\operatorname{Hecke}}(\delta_{1,\operatorname{Gr}_G})$. It remains to show that

$$(7.7) \qquad \operatorname{Fun}(\operatorname{Fl}^{\check{G}}_1) \to \operatorname{Hom}_{\operatorname{D}(\operatorname{Gr}_G)_{\operatorname{crit}}\!-\!\operatorname{mod}^{I^0}}\left(\delta_{1,\operatorname{Gr}_G}, \mathcal{W}_1\right)$$

is an isomorphism and

$$(7.8) \qquad \operatorname{Ext}^1_{\operatorname{D}(\operatorname{Gr}_G)_{\operatorname{crit}}\!-\!\operatorname{mod}^{I^0}}\left(\delta_{1,\operatorname{Gr}_G}, \mathcal{W}_1\right) = 0,$$

where in both formulas $\mathcal{W}_1$ appears as an object of $\operatorname{D}(\operatorname{Gr}_G)_{\operatorname{crit}}\!-\!\operatorname{mod}^{I^0}$ via the tautological forgetful functor.

Recall (see Sect. 5.7) that $\mathcal{O}_{\operatorname{Fl}^{\check{G}}_1}$ can be written as a filtered direct limit

$$\varinjlim_{\check{\mu}\in\check{\Lambda}^+} \mathcal{L}^{\check{\mu}}_{\operatorname{Fl}^{\check{G}}} \otimes \mathfrak{l}^{-\check{\mu}},$$

where $\mathfrak{l}^{\check{\mu}}$ denotes the $\check{B}$-stable line in $V^{\check{\mu}}$, and $\mathfrak{l}^{-\check{\mu}} \subset (V^{\check{\mu}})^*$ is its dual.

Hence, $\mathcal{W}_1$ can also be written down as a filtered direct limit

$$\varinjlim_{\check{\mu}\in\check{\Lambda}^+} \operatorname{Ind}^{\operatorname{Hecke}}(j_{\check{\mu},\operatorname{Gr}_G,*}) \otimes \mathfrak{l}^{-\check{\mu}}.$$

Recall that $\operatorname{Ind}^{\operatorname{Hecke}}(\mathcal{F}) \simeq \mathcal{F} \underset{G[[t]]}{\star} \mathcal{F}_{R(\check{G})}$. We have $\mathcal{F}_{R(\check{G})} \simeq \underset{\check{\nu}}{\oplus}\mathcal{F}_{(V^{\check{\nu}})^*} \otimes V^{\check{\nu}}$. Now the isomorphism in (7.7) follows from the fact that

$$
\begin{cases}
\operatorname{Hom}(\delta_{1,\operatorname{Gr}_G}, j_{\check{\mu},\operatorname{Gr}_G,*} \underset{G[[t]]}{\star} \mathcal{F}_{(V^{\check{\nu}})^*}) = 0, \ \check{\mu} \neq \check{\nu}, \\[2mm]
\operatorname{Hom}(\delta_{1,\operatorname{Gr}_G}, j_{\check{\mu},\operatorname{Gr}_G,*} \underset{G[[t]]}{\star} \mathcal{F}_{(V^{\check{\nu}})^*}) \simeq \mathbb{C}, \ \check{\mu} = \check{\nu}.
\end{cases}
$$

The vanishing of (7.8) follows from the fact that

$$\operatorname{Ext}^1(\delta_{1,\operatorname{Gr}_G}, j_{\check{\mu},\operatorname{Gr}_G,*} \underset{G[[t]]}{\star} \mathcal{F}_{(V^{\check{\nu}})^*}) \simeq \operatorname{Ext}^1(\operatorname{IC}_{\overline{\operatorname{Gr}}_G^{\check{\nu}}}, j_{\check{\mu},\operatorname{Gr}_G,*}) = 0,$$

by the parity vanishing of IC-stalks on Gr_G.

8. Lie algebroids and the renormalized enveloping algebra

In this section we will study the interaction between the isomorphism $\operatorname{map}_{\operatorname{geom}}$ and certain canonical Lie algebroids defined on the two sides of this isomorphism.

8.1. Let $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$ denote the conormal to $\operatorname{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}})$ inside $\operatorname{Spec}(\mathfrak{Z}_{\mathfrak{g}})$, equipped with a natural topology. Recall (see [BD], Sect. 3.6 or [FG2], Sect. 7.4) that $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$ has a natural structure of Lie algebroid over $\operatorname{Spec}(\mathfrak{Z}_{\mathfrak{g}})$.

On the other hand, let us recall the groupoid $\operatorname{Isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}$ over $\operatorname{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}})$, see (4.1), and let $\operatorname{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}$ be the corresponding Lie algebroid.

According to [BD], Theorem 3.6.7, there exists a canonical morphism (in fact, an isomorphism) of Lie algebroids:

$$(8.1) \qquad\qquad\qquad v_{\operatorname{geom}} : N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}} \to \operatorname{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}.$$

Below we will recall the definition of this map. Let us note that both the Lie algebroid structure on $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$ and the morphism v_{geom} depend on an additional choice of a one-parameter deformation $\kappa_{\hbar}$ of the level away from the critical value.

8.2. **The renormalized enveloping algebra.** Let $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$ be the renormalized universal enveloping algebra at the critical level, which is defined in [BD], Sect. 5.6 (see also [FG2], Sect. 7.4 for a review). It admits a natural filtration, with the 0-th term $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_0$ isomorphic to the topological algebra

$$U^{\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}}) := \widetilde{U}_{\operatorname{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}},$$

responsible for the category $\widehat{\mathfrak{g}}_{\operatorname{crit}}-\operatorname{mod}_{\operatorname{reg}}$ with its tautological forgetful functor to Vect. The first associated graded quotient $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_1/U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_0$ is isomorphic to

$$U^{\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}{\widehat{\otimes}} N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}.$$

Note that $\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}$ is a subalgebra in $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$, but it is no longer central.

Let us recall some basic constructions related to $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$.

1. Let $\mathcal{M}_1, \mathcal{M}_2$ be two objects of $\widehat{\mathfrak{g}}_{\operatorname{crit}}-\operatorname{mod}_{\operatorname{reg}}$, on which the action of $U^{\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$ has been extended to an action of $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$. Then $\operatorname{Hom}_{\widehat{\mathfrak{g}}_{\operatorname{crit}}}(\mathcal{M}_1, \mathcal{M}_2)$ acquires a natural action of $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$ via $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}} \hookrightarrow U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_1/U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_0$.

2. Let $\mathcal{M}$ be a $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$-module, and let $\mathcal{L}$ be a $\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}$ module that carries a compatible action of $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$. Then $\mathcal{M} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}{\otimes} \mathcal{L}$ is naturally a $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$-module.

Indeed, we define the action of

$$U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_1 \underset{U^{\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}}{\widehat{\otimes}} N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}}{\times} N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}} \subset U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$$

to be the sum of the given $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})_1$-action on $\mathcal{M}$ and the $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\operatorname{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$-action on $\mathcal{L}$. From the relations that realize $U^{\operatorname{ren},\operatorname{reg}}(\widehat{\mathfrak{g}}_{\operatorname{crit}})$ as a quotient of the universal enveloping algebra of

$U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_1$, it follows that the above definition extends to a well-defined $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action on $\mathcal{M} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}$.

3. Let $\mathcal{M}_\hbar$ be a flat $\mathbb{C}[\hbar]$-family of $\widehat{\mathfrak{g}}_{\kappa_\hbar}$-modules, such that $\mathcal{M}_{\mathrm{crit}} := \mathcal{M}_\hbar / \hbar \cdot \mathcal{M}$ belongs to $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –mod$_{\mathrm{reg}}$, then $\mathcal{M}_{\mathrm{crit}}$ acquires an action of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

The prime example of this situation is $\mathcal{M}_\hbar = \mathbb{V}_\hbar$. In this case we obtain the canonical $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action on $\mathbb{V}_{\mathrm{crit}}$.

4. The adjoint action of $G((t))$ on $U^{\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ extends naturally to an action on $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$. Hence, the category $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ –mod is acted on by critically twisted D-modules on $G((t))$ by convolutions (see [FG2], Sect. 22).

In particular, if $\mathcal{M}$ is a $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-module, which is $\mathfrak{g}[[t]]$-integrable as a $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module, and $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$ –mod, we obtain a well-defined object $\mathcal{F} \underset{G[[t]]}{\star} \mathcal{M}$ in the derived category of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-modules. Applying this to $\mathcal{M} = \mathbb{V}_{\mathrm{crit}}$, we recover the canonical $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action on $\mathcal{F} \underset{G[[t]]}{\star} \mathbb{V}_{\mathrm{crit}} \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F})$.

8.3. We are now ready to construct the map v_{geom}. By definition, $\mathsf{isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$ is the Atiyah algebroid of the $\check{G}$-torsor $\mathcal{P}_{\check{G}, \mathfrak{Z}}$. Hence, to specify v_{geom} we need to make $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} / \mathfrak{Z}_{\mathfrak{g}}}$ act on $\mathcal{V}_{\mathfrak{Z}}$ for every $V \in \mathrm{Rep}(\check{G})$ in a way compatible with tensor products.

However, by construction, $\mathcal{V}_{\mathfrak{Z}} \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}_{\mathrm{crit}}, \Gamma(\mathrm{Gr}_G, \mathcal{F}_V))$, and the required action follows from **1** and **4** above. In other words, the isomorphism $\Gamma(\mathrm{Gr}_G, \mathcal{F}_V) \simeq \mathbb{V}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}_{\mathfrak{Z}}$ is compatible with the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions, where on the RHS it is given via the construction of **2** above. The compatibility with tensor products follows from the compatibility between the constructions in **2** and **4**.

We will now recall the definition of another map

$$(8.2) \qquad v_{\mathrm{alg}} : N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} / \mathfrak{Z}_{\mathfrak{g}}} \to \mathsf{isom}_{\check{G}, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}},$$

following [BD], Sect. 3.7.13.

We recall that the isomorphism of topological algebras

$$(8.3) \qquad \mathrm{map}_{\mathrm{alg}} : \mathfrak{Z}_{\mathfrak{g}} \simeq \mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times))$$

respects the Poisson structures, where on the LHS the Poisson bracket is defined via the Drinfeld-Sokolov reduction (see [FG2], Sect. 4 for a review), using the $\check{G}$-invariant form $\check{\kappa}$ on $\check{\mathfrak{g}}$, corresponds to the deformation $\kappa_\hbar$ of the level off the critical value (see Sect. 8.9 for the precise formulation).

Thus, the map $\mathrm{map}_{\mathrm{alg}}$ induces an isomorphism of algebroids

$$N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} / \mathfrak{Z}_{\mathfrak{g}}} \simeq \mathrm{map}^*_{\mathrm{alg}}\big(N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} / \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}\big).$$

Combining this with the isomorphism

$$(8.4) \qquad N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} / \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)} \simeq \mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$$

(see, e.g., [FG2], Sect. 4.4, where the latter is explained), we obtain the map of (8.2). By construction, the map v_{alg} is an isomorphism.

Our present goal of this section is to prove the following:

Theorem 8.4. *The maps v_{geom} and v_{alg} coincide.*

The above theorem has been proved in [BD], Proposition 3.5.13, simultaneously with Theorem 1.4. Namely, in *loc. cit.* it was shown that the algebroid $\mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ over $\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}})$ does not admit non-trivial automorphisms. We will give a constructive proof of this result, which will occupy the rest of this section.

8.5.   The algebroid $\mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ acts naturally on the scheme $\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}$, and it is easy to see that the corresponding action on $\mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}})$ is faithful. Hence, to prove Theorem 8.4, it suffices to see that the two resulting actions of $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$ on $\mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}})$– one via υ_{geom} and another via υ_{alg}– coincide.

Consider the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action on $\mathbb{W}_{\mathrm{crit},0}$, corresponding to the $\hbar$-family $\mathbb{W}_{\hbar,0}$. Recall that we have an isomorphism
$$\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq \mathrm{End}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{W}_{\mathrm{crit},0}).$$
Hence, from Sect. 8.2(**1**) we obtain an action of $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$ on $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$.

Recall now isomorphism
$$\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq \mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}).$$
Thus, to prove Theorem 8.4 it suffices to prove the following two assertions:

Proposition 8.6. *The above action of $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$ on $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$ goes over under the map υ_{geom} to the natural action of $\mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ on $\mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}})$.*

Proposition 8.7. *The above action of $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$ on $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$ goes over under the map υ_{alg} to the natural action of $\mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ on $\mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}})$.*

8.8. **Proof of Proposition 8.6.** Let $\mathrm{Spec}(\mathcal{B})$ be an affine scheme over $\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}})$, endowed with a map to $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$. Note that such a data is specified by a $\check{H}$-torsor $\{\check{\lambda} \mapsto \mathcal{L}_\mathcal{B}^{\check{\lambda}}\}$ and a collection of maps
$$\kappa_\mathcal{B}^{-,\check{\lambda}} : \mathcal{V}_\mathfrak{Z}^{\check{\lambda}} \to \mathcal{L}_\mathcal{B}^{\check{\lambda}}, \ \check{\lambda} \in \check{\Lambda}^+,$$
satisfying the Plücker equations.

Let $\mathfrak{f}$ be a Lie algebroid over $\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}})$, endowed with a map $\upsilon : \mathfrak{f} \to \mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$. In particular, for every $V \in \mathrm{Rep}(\check{G})$ we obtain an $\mathfrak{f}$-action on $\mathcal{V}_\mathfrak{Z}$.

Suppose, in addition, that we are given an action of $\mathfrak{f}$ on $\mathcal{B}$. Then these data are compatible with the natural $\mathsf{isom}_{\check{G},\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$-action on $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$ if and only if the following holds:

For every $\check{\lambda} \in \check{\Lambda}^+$ there exists an $\mathfrak{f}$-action on the line bundle $\mathcal{L}_\mathcal{B}^{\check{\lambda}}$, such that the map $\kappa_\mathcal{B}^{-,\check{\lambda}}$ is compatible with the $\mathfrak{f}$-actions. (Note that such an action on $\mathcal{L}_\mathcal{B}^{\check{\lambda}}$ is *a priori* unique, since the induced maps $\mathcal{V}_\mathfrak{Z}^{\check{\lambda}} \underset{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}{\otimes} \mathcal{B} \to \mathcal{L}_\mathcal{B}^{\check{\lambda}}$ are surjective.)

Let us apply this for $\mathcal{B} = \mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$ and $\mathfrak{f} = N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$. Recall that the map $\mathrm{Spec}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}) \to \mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$ corresponds to the collection of line bundles $\mathcal{L}_\mathfrak{H}^{\check{\lambda}}$ and the maps $\kappa_\mathfrak{H}^{-,\check{\lambda}}$ of (2.7).

Consider the Wakimoto module $\mathbb{W}_{\mathrm{crit},0}$, endowed with the natural $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action. By construction, the map $\phi : \mathbb{V}_{\mathrm{crit}} \to \mathbb{W}_{\mathrm{crit},0}$ deforms off the critical level to a map $\phi_\hbar : \mathbb{V}_\hbar \to \mathbb{W}_{\hbar,0}$; hence the map ϕ is compatible with the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

Consider now $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$. From Sect. 8.2(**4**), we obtain that $j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$ also carries an action of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, and the map
$$\Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}}) \to j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}$$

of (2.5) is compatible with the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

From Sect. 8.2(**1**) We obtain that $\mathcal{L}_{\check{\mathfrak{H}}}^{\check{\lambda}} \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{W}_{\mathrm{crit},0}, j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0})$ acquires an action of $N^*_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{z}_{\mathfrak{g}}}$, and the map $\kappa_{\check{\mathfrak{H}}}^{-,\check{\lambda}}$, which equals

$$
\mathcal{V}_{\mathfrak{z}}^{\check{\lambda}} \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}\left(\mathbb{V}_{\mathrm{crit}}, \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\lambda}}})\right) \rightarrow
$$
$$
\rightarrow \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}\left(\mathbb{V}_{\mathrm{crit}}, j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}\right) \overset{\sim}{\leftarrow} \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}\left(\mathbb{W}_{\mathrm{crit},0}, j_{\check{\lambda},*} \star \mathbb{W}_{\mathrm{crit},0}\right) \simeq \mathcal{L}_{\check{\mathfrak{H}}}^{\check{\lambda}},
$$

is compatible with the $N^*_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{z}_{\mathfrak{g}}}$-actions, which is what we had to show.

8.9. **The Poisson structure on Miura opers.** In order to prove Proposition 8.7 we need to make a digression and discuss the Poisson structure on the space of Miura opers.

Let $\frac{d\kappa_\hbar}{d\hbar}|_{\kappa_{\mathrm{crit}}}$ be the derivative of $\kappa_\hbar$ at κ_{crit}, which is, by definition, a non-degenerate G-invariant bilinear form on $\mathfrak{g}$. We can view it as a non-degenerate W-invariant bilinear form on $\mathfrak{h}$, which, in turn, can be thought of as a non-degenerate W-invariant bilinear form on $\check{\mathfrak{h}}$, or as a $\check{G}$-invariant bilinear form on $\check{\mathfrak{g}}$. We will denote the latter by $\check{\kappa}$.

Recall the scheme $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$. Proceeding as in [FG2], Sect. 4.3, (with $\check{G}$ replaced by $\check{H}$), using the form $\check{\kappa}$, we define a Poisson structure on $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$.

Let $\mathsf{Isom}_{\check{H}, \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)}$ denote the natural groupoid acting on $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$, whose fiber over two points $\widehat{\chi}_1, \widetilde{\chi}_2 \in \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$ is the ind-scheme of automorphisms of the $\check{H}$-torsor $\omega_{\mathcal{D}^\times}^{\check{\rho}}$ over $\mathcal{D}^\times$, that transforms connection $\widehat{\chi}_1$ to $\widehat{\chi}_2$. Let $\mathsf{isom}_{\check{H}, \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)}$ denote the corresponding Lie algebroid on $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$.

As in [FG2], Sect. 4.3, one easily shows that there exists a canonical isomorphism of Lie algebroids

$$
(8.5) \qquad\qquad \Omega^1(\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)) \simeq \mathsf{isom}_{\check{H}, \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)}.
$$

Recall that the Poisson structure on $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)$ also depended on the form $\check{\kappa}$ on $\check{\mathfrak{g}}$. It is a straightforward calculation that the map $\mathrm{MT} : \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \rightarrow \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)$ is compatible with the Poisson structures. Therefore, the pull-back $\mathrm{MT}^*\left(\Omega^1(\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times))\right)$ acquires a structure of Lie algebroid and we have a homomorphism of Lie algebroids over $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$:

$$
\mathrm{MT}^*\left(\Omega^1(\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times))\right) \rightarrow \Omega^1(\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)).
$$

Recall now the Lie algebroid $\mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}$ over $\mathrm{Op}_{\check{\mathfrak{g}}}(D^\times)$. By the definition of generic Miura opers, this algebroid acts on the ind-scheme $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}(\mathcal{D}^\times)$, which, as we know, identifies with $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$.

Hence, the pull-back $\mathrm{MT}^*\left(\mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}\right)$ acquires the structure of a Lie algebroid on the ind-scheme $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$. In fact, the anchor map $\mathrm{MT}^*\left(\mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}\right) \rightarrow T(\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho))$ naturally factors through a map

$$
\mathrm{MT}^*\left(\mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}\right) \rightarrow \mathsf{isom}_{\check{H}, \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)}.
$$

Finally, recall that we have a canonical isomorphism of algebroids on $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)$ (see [FG2], Sect. 4.3):

$$
(8.6) \qquad\qquad \Omega^1(\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)) \simeq \mathsf{isom}_{\check{G}, \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}.
$$

The following results from the constructions:

Lemma 8.10. *The following diagram is commutative*

$$
\begin{array}{ccc}
\mathrm{MT}^*\big(\Omega^1(\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times))\big) & \xrightarrow{\ \mathrm{MT}^*\ (8.6)\ } & \mathrm{MT}^*\big(\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}\big) \\
\downarrow & & \downarrow \\
\Omega^1(\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)) & \xrightarrow{\ \ (8.5)\ \ } & \mathsf{isom}_{\check{H},\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)}.
\end{array}
$$

As the result, we obtain:

Corollary 8.11. *The action of* $N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}/\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}$ *on* $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho)$, *resulting from the Poisson structure on on* $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$ *and the map* $\mathrm{MT}: \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho) \to \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$, *identifies via*

$$
N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}/\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)} \overset{(8.4)}{\simeq} \mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}} \ and \ \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho) \overset{(2.3)}{\simeq} \mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}^{\mathrm{reg}}
$$

with the canonical action of $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$ *on* $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}^{\mathrm{reg}}$.

8.12. **Proof of Proposition 8.7.** The deformation $\kappa_\hbar$ of the level defines a non-commutative deformation $\widehat{\mathfrak{H}}_{\kappa_\hbar}$ of $\widehat{\mathfrak{H}}_{\mathrm{crit}}$. This endows $\widehat{\mathfrak{H}}_{\mathrm{crit}}$ with a Poisson structure. Moreover, as we shall see in Sect. 9.8, the map $\varphi: \mathfrak{Z}_\mathfrak{g} \to \widehat{\mathfrak{H}}_{\mathrm{crit}}$ is Poisson and has the following property:

The resulting action of $N^*_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}$ on $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}}$ coincides with the one coming from the isomorphism $\mathfrak{H}_{\mathrm{crit}}^{\mathrm{reg}} \simeq \mathrm{End}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{W}_{\mathrm{crit},0})$ via the construction of Sect. 8.2(**1**).

In addition, it is easy to see that the isomorphism, induced by $\mathrm{map}_{\mathrm{alg}}^M$,

$$
\widehat{\mathfrak{H}}_{\mathrm{crit}} \to \mathrm{Fun}(\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho))
$$

respects the Poisson structures. Hence, the same is true for $\mathrm{map}_{\mathrm{geom}}^M = \tau \circ \mathrm{map}_{\mathrm{alg}}^M$.

Thus, we obtain a commutative diagram of Poisson ind-schemes:

$$
\begin{array}{ccc}
\mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}}^M = \tau \circ \mathrm{map}_{\mathrm{alg}}^M\ } & \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \\
\varphi \downarrow & & \mathrm{MT} \downarrow \\
\mathrm{Spec}(\mathfrak{Z}_\mathfrak{g}) & \xrightarrow{\ \mathrm{map}_{\mathrm{geom}} = \mathrm{map}_{\mathrm{alg}}\ } & \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times),
\end{array}
$$

and the assertion of the proposition follows from Corollary 8.11.

9. LIE ALGEBROIDS AND WAKIMOTO MODULES

9.1. **Algebroids acting on categories.** Let $\mathbf{A}$ be a commutative algebra, and let $\mathfrak{f}$ be a (topological) Lie algebroid over $\mathbf{A}$. We shall assume that as an $\mathbf{A}$-module, $\mathfrak{f}$ is the dual of a discrete projective $\mathbf{A}$-module, denoted $\mathfrak{f}^*$.

Let $\mathcal{C}$ be a $\mathbf{A}$-linear category, i.e., $\mathbf{A}$ acts by endomorphisms on every object of $\mathcal{C}$ in a functorial way. In this case one can introduce the notion of *action of* $\mathfrak{f}$ *on* $\mathcal{C}$. This is, by definition, the same as an action on $\mathcal{C}$ of the formal groupoid $\mathfrak{F}$, corresponding to $\mathfrak{f}$. (The latter notion is spelled out explicitly for groupoids in [Ga], and for group ind-schemes in [FG2], Sect. 22; the generalization to the case of arbitrary formal groupoids is straightforward.)

A basic example of this situation is when $\mathcal{C}$ is taken to be the category of $\mathbf{A}$-modules.

An action of $\mathfrak{f}$ on $\mathcal{C}$ is specified by the following data. For every object $\mathcal{M} \in \mathcal{C}$ there must be a functorially assigned extension

$$
0 \to \mathcal{M} \underset{\mathbf{A}}{\otimes} \mathfrak{f}^* \to \mathrm{act}_{\mathfrak{f}}^*(\mathcal{M}) \to \mathcal{M} \to 0,
$$

such that for $\mathbf{a} \in \mathbf{A}$, the difference between its action on $\mathcal{M}_{\mathfrak{f}}$ as an object of $\mathcal{C}$, and the action coming from the functoriality of $\mathrm{act}^*_{\mathfrak{f}}$ and the action of $\mathbf{a}$ on $\mathcal{M}$, is the map

$$\mathcal{M} \mapsto \mathcal{M} \underset{\mathbf{A}}{\otimes} \mathfrak{f}^*$$

given by the image of $d(\mathbf{a})$ under the dual of the anchor map $\Omega^1(\mathbf{A}) \to \mathfrak{f}^*$.

The functor $\mathrm{act}^*_{\mathfrak{f}}$ must, in addition, be equipped with a *Lie constraint*, which is a natural transformation that relates the iteration $\mathrm{act}^*_{\mathfrak{f}} \circ \mathrm{act}^*_{\mathfrak{f}}$ with the Lie bracket on $\mathfrak{f}$. This natural transformation must satisfy an identity for the 3-fold iteration. We will not spell this out explicitly.

Given an action of $\mathfrak{f}$ on $\mathcal{C}$, we say that an object $\mathcal{M} \in \mathcal{C}$ if $\mathfrak{f}$-equivariant if we are given a splitting $\mathrm{act}^*_{\mathfrak{f}}(\mathcal{M}) \leftarrow \mathcal{M}$, compatible with the Lie constraint.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two categories, acted on by $\mathfrak{f}$, there is an evident notion of functor between them, compatible with the $\mathfrak{f}$-actions.

9.2. Let us give a typical example of how actions of Lie algebroids arise in practice. Let $\mathbf{B}$ be a topological $\mathbf{A}$-algebra. Suppose we are given a topological Lie algebroid $\mathfrak{f}'$ over $\mathbf{A}$ that fits into a diagram of Lie algebras

$$
\begin{array}{c}
\mathbf{B} \\
\uparrow \\
0 \longrightarrow \mathfrak{f}'' \longrightarrow \mathfrak{f}' \longrightarrow \mathfrak{f} \longrightarrow 0,
\end{array}
$$

and we are given a continuous action of $\mathfrak{f}'$ on $\mathbf{B}$ by derivations, which extends the action of $\mathfrak{f}$ on $\mathbf{A}$ and $\mathfrak{f}''$ on $\mathbf{B}$.

Note that in this case we can form a topological associative enveloping algebra, call it $\mathbf{B}^{\mathrm{ren}}$, which is universal with respect to the property that $\mathbf{B}$ maps to it, as an associative subalgebra, and $\mathfrak{f}'$, as a Lie algebra, in a compatible way.

Let $\mathbf{B}$–mod denote the category of (discrete, continuous) $\mathbf{B}$-modules.

Lemma 9.3.

(1) *Under the above circumstances we have a canonical action of $\mathfrak{f}$ on the category $\mathbf{B}$ –mod.*

(2) *Specifying an $\mathfrak{f}$-equivariant structure on $\mathcal{M} \in \mathbf{B}$ –mod is equivalent to extending the $\mathbf{B}$-action on it to a $\mathbf{B}^{\mathrm{ren}}$-action.*

We will use the following general construction.

Let $\mathbf{B}_{\hbar}$ be a flat $\mathbb{C}[\hbar]$ family of topological associative algebras; set $\mathbf{B}_0 = \mathbf{B}_{\hbar}/\hbar \cdot \mathbf{B}_{\hbar}$, and let $\mathbf{Z}$ be the center of $\mathbf{B}_0$. As in [BD], Sect. 5.6, $\mathbf{Z}$ acquires a natural Poisson bracket.

Let $\mathbf{A} \subset \mathbf{Z}$ be a (closed) subalgebra, closed under the Poisson bracket, and let $\mathbf{I} \subset \mathbf{A}$ be an open Poisson ideal; in particular $\mathbf{A}^{\mathrm{reg}} := \mathbf{A}/\mathbf{I}$ is a discrete Poisson algebra. We will assume, in addition, that the multiplication map $\mathbf{I} \widehat{\underset{\mathbf{A}}{\otimes}} \mathbf{I} \to \mathbf{I}$ is a closed embedding, and that $\mathbf{I}/\mathbf{I}^2$ is the dual of a projective $\mathbf{A}^{\mathrm{reg}}$-module. Then $\mathbf{I}/\mathbf{I}^2$ is naturally a topological Lie algebroid over $\mathbf{A}^{\mathrm{reg}}$. Denote $\mathbf{B}^{\mathrm{reg}} := \mathbf{B}_0/\mathbf{B}_0 \cdot \mathbf{I}$.

Proposition-Construction 9.4.

(1) *Under the above circumstances, the $\mathbf{A}^{\mathrm{reg}}$-linear category $\mathbf{B}^{\mathrm{reg}}$ –mod carries a natural action of $\mathbf{I}/\mathbf{I}^2$.*

(2) *If $\mathcal{M}_{\hbar}$ is a flat family of $\mathbf{B}_{\hbar}$-modules, such that the action of $\mathbf{I}$ on $\mathcal{M}_0 := \mathcal{M}/\hbar \cdot \mathcal{M}$ is zero, then, as an object of $\mathbf{B}^{\mathrm{reg}}$ –mod, $\mathcal{M}_0$ is naturally $\mathbf{I}/\mathbf{I}^2$-equivariant.*

Proof. Let $\mathbf{B}_\hbar^\sharp$ be the $\mathbb{C}[\hbar]$-submodule in the localization of $\mathbf{B}_\hbar$ with respect to $\hbar$, formed by expressions $\frac{\mathbf{b}_\hbar}{\hbar}$, where $\mathbf{b}_0 := \mathbf{b}_\hbar \mod \hbar \in \mathbf{I}$. Set $\mathbf{B}^\sharp = \mathbf{B}_\hbar^\sharp/\hbar \cdot \mathbf{B}_\hbar^\sharp$. It fits into a short exact sequence

$$0 \to \mathbf{B}_0/\mathbf{I} \to \mathbf{B}^\sharp \to \mathbf{I} \to 0.$$

The algebra $\mathbf{A}$ acts naturally on $\mathbf{B}^\sharp$ by left multiplication. Set $\mathbf{B}^\flat := \mathbf{B}^\sharp/\mathbf{I} \cdot \mathbf{B}^\sharp$.

By the assumption on $\mathbf{I}$, we have a short exact sequence:

$$(9.1) \qquad\qquad 0 \to \mathbf{B}^{\mathrm{reg}} \to \mathbf{B}^\flat \to \mathbf{I}/\mathbf{I}^2 \to 0.$$

This brings us to the context of Lemma 9.3 with $\mathfrak{f} = \mathbf{I}/\mathbf{I}^2$ and $\mathfrak{f}' = \mathbf{B}^\flat$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will apply the above Proposition-Construction to topological associative algebras, attached to chiral algebras at a given point of a curve. We should remark that one can avoid dealing with topological associative algebras and instead of using Proposition-Construction 9.4, one can work directly with chiral algebras, as was done in [FG1], Sect. 4:

Let $\mathcal{B}_\hbar$ be a $\mathbb{C}[\hbar]$-flat family of chiral algebras. Denote $\mathcal{B}_0 = \mathcal{B}_\hbar/\hbar \cdot \mathcal{B}_\hbar$, and let $\mathfrak{z}(\mathcal{B}_0)$ be its center. It is known that $\mathfrak{z}(\mathcal{B}_0)$ acquires a natural coisson structure.

Let $\mathcal{A}$ be a (commutative) chiral subalgebra of $\mathfrak{z}(\mathcal{B}_0)$, and we will assume that $\mathcal{A}$ is stable under the coisson bracket. We will assume that $\mathcal{A}$ is smooth; in particular, the module $\Omega^1(\mathcal{A})$ is projective as a $\mathcal{A} \otimes \mathrm{D}_X$-module.

Let $\mathcal{B}_0 -\mathrm{mod}_{\mathrm{reg}}$ denote the full subcategory in the category of chiral $\mathcal{B}_0$-modules, supported at x, on which the Lie-* action of $\mathcal{A}$ is zero. By definition, this is a $\mathcal{A}_x$-linear category, where $\mathcal{A}_x$ denotes the fiber of $\mathcal{A}$ at x.

We claim that under the above circumstances, we have a naturally defined action of

$$N^*_{\mathcal{A}_x/\widehat{\mathcal{A}}_x} \simeq H^0_{DR}(\mathcal{D}_x, \Omega^1(\mathcal{A}))$$

on the category $\mathcal{B}_0 -\mathrm{mod}_{\mathrm{reg}}$.

Indeed, by proceeding along the lines of Proposition-Construction 9.4, one can produce a Lie-* algebroid over $\mathcal{A}$, denoted $\mathcal{B}^\flat$, that fits into a short exact sequence

$$0 \to \mathcal{B}_0/\mathcal{A} \to \mathcal{B}^\flat \to \Omega^1(\mathcal{A}) \to 0.$$

We define the Lie-* algebroid $\mathfrak{f}'$ to be $H^0_{DR}(\mathcal{D}_x, \mathcal{B}^\flat)$, and then apply Lemma 9.3.

9.5. Let us return to the setting of the affine algebra at the critical level. We will consider various categories over the commutative algebra $\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}$.

Let us consider several examples of $\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}$-linear categories, equipped with an action of the algebroid $\mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$.

1. Let $\mathcal{C}$ be $\mathrm{QCoh}(\mathrm{Fl}_\mathfrak{Z}^{\check{G}})$. The action of the groupoid $\mathsf{Isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ on $\mathrm{Fl}_\mathfrak{Z}^{\check{G}}$ defines a natural action of $\mathsf{Isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$ on $\mathrm{QCoh}(\mathrm{Fl}_\mathfrak{Z}^{\check{G}})$. Hence, we obtain the action on $\mathrm{QCoh}(\mathrm{Fl}_\mathfrak{Z}^{\check{G}})$ of its algebroid, which is by definition, $\mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$.

Evidently, for every $w \in W$ the subcategory $\mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ also carries an action of $\mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$. (But, of course, this action does not come from an action of $\mathsf{Isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$.)

2. Let us now take $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_\mathfrak{Z}}_{\mathrm{crit}} -\mathrm{mod}$. It carries a natural action of the groupoid $\mathsf{Isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$, and hence also of the algebroid $\mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$.

Let us write down this action more explicitly. For $(\mathcal{F}, \{\alpha_V\}) \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_\mathfrak{Z}}_{\mathrm{crit}} -\mathrm{mod}$ we define the corresponding extension $\mathrm{act}^*_{\mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}}(\mathcal{F}, \{\alpha_V\})$ as follows.

The underlying D-module is the trivial extension

$$\mathcal{F} \underset{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} (\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}})^* \oplus \mathcal{F},$$

with the $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-action twisted by map

$$\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \overset{d}{\to} \Omega^1(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to (\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}})^*.$$

The corresponding isomorphisms α_V are obtained from the original ones by adding the term involving the $\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$-action on $\mathcal{V}_{\mathfrak{z}}$.

By construction, the functor $\mathsf{E}^{\mathfrak{z}} : \mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}) \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{z}}}$ –mod, defined in Sect. 4.3, respects the action of the groupoid $\mathsf{Isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$, and hence of the algebroid $\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$.

One can see the latter compatibility explicitly for quasi-coherent sheaves $\mathcal{T}$ that are direct images from affine open subsets of $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$, using the explicit description of $\mathsf{E}^{\mathfrak{z}}(\mathcal{T})$ in this case as a co-equalizer, see Sect. 4.3.

3. Finally, let us take $\mathcal{C} = \widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}$. This example fits into the framework of (the chiral algebra version of) Proposition-Construction 9.4 for the $\mathbb{C}[\hbar]$-family $\mathcal{B}_\hbar := \mathcal{A}_{\mathfrak{g}, \kappa_\hbar, X}$, and $\mathcal{A} = \mathfrak{z}(\mathcal{A}_{\mathfrak{g}, \mathrm{crit}, X})$.

We obtain an action of $N_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{z}_{\mathfrak{g}}}^*$ on $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}$, and hence also of $\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$, via the identification υ_{geom}.

Note that the topological associative algebra $\mathbf{B}^{\mathrm{ren}}$ identifies, by definition, with the renormalized algebra $U^{\mathrm{ren}, \mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

As was shown in [FG4], Sect. 2.9., the functor

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{z}}} : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{z}}} \text{ –mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}} \text{ –mod}_{\mathrm{reg}}$$

is compatible with the $\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$-actions. From this we obtain that the functor

$$\mathsf{G} = \Gamma^{\mathrm{Hecke}_{\mathfrak{z}}} \circ \mathsf{E} : \mathrm{QCoh}(\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}) \to \widehat{\mathfrak{g}}_{\mathrm{crit}} \text{ –mod}_{\mathrm{reg}}$$

is also compatible with the $\mathsf{isom}_{\check{G}, \mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}}$-actions.

9.6. Recall the topological algebra $\widehat{\mathfrak{H}}_{\mathrm{crit}}$ and consider the category $\left(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\right)$ –mod, from which, we recall, we had the Wakimoto functor to the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}$. We will establish the following:

Proposition-Construction 9.7.

(1) *The category* $\left(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\right)$ –mod *carries a natural action of* $N_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{z}_{\mathfrak{g}}}^*$.

(2) *The functor* $\mathbb{W} : \left(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\right)$ –mod $\to \widehat{\mathfrak{g}}_{\mathrm{crit}}$ –$\mathrm{mod}_{\mathrm{reg}}$ *is compatible with the* $N_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{z}_{\mathfrak{g}}}^*$ *-actions.*

The rest of this subsection is devoted to the proof of this Proposition 9.7. To carry it out we need to review the framework in which Wakimoto modules were defined. We will follow the conventions of [FG2], Sect. 10.

For every level κ we have the Heisenberg chiral algebra $\mathfrak{H}_{\kappa, X}$, and a chiral algebra, denoted $\mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\kappa, X}$, which is isomorphic to the tensor product

$$\mathfrak{D}^{ch}(N)_X \otimes \mathfrak{H}_{\kappa, X},$$

where $\mathfrak{D}^{ch}(N)_X$ is the chiral algebra of differential operators on N, which is independent of the level.

In particular, for every $\mathfrak{H}_{\kappa,X}$-module $\mathfrak{T}$, supported at $x \in X$, we can consider the $\mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\kappa,X}$-module $\mathfrak{D}^{ch}(N)_x \otimes \mathfrak{T}$, where $\mathfrak{D}^{ch}(N)_x$ denotes the vacuum module of $\mathfrak{D}^{ch}(N)_X$ at x.

We have a canonical bosonization map $\mathfrak{l}_\kappa : \mathcal{A}_{\mathfrak{g},\kappa,X} \to \mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\kappa,X}$. The Wakimoto functor associates at $\mathfrak{T}$ as above the module $\mathfrak{D}^{ch}(N)_x \otimes \mathfrak{T}$, regarded as a $\mathcal{A}_{\mathfrak{g},\kappa,X}$-module via $\mathfrak{l}_\kappa$.

Let us take $\kappa = \kappa_{\mathrm{crit}}$. In this case $\mathfrak{H}_{\mathrm{crit},X}$ equals the center of $\mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\mathrm{crit},X}$. Moreover, we have the following commutative diagram:

(9.2)
$$
\begin{array}{ccc}
\mathfrak{z}_{\mathfrak{g},X} & \longrightarrow & \mathcal{A}_{\mathfrak{g},\mathrm{crit},X} \\
\varphi \downarrow & & \mathfrak{l}_{\mathrm{crit}} \downarrow \\
\mathfrak{H}_{\mathrm{crit},X} & \longrightarrow & \mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\mathrm{crit},X}.
\end{array}
$$

We apply the construction of Sect. 9.2 for $\mathcal{B}_\hbar$ being each of the following three chiral algebras: $\mathcal{A}_{\mathfrak{g},\kappa_\hbar,X}$, $\mathfrak{D}^{ch}(\overset{\circ}{G/B})_{\kappa_\hbar,X}$ and $\mathfrak{H}_{\kappa_\hbar,X}$. (In what follows, we will replace the subscript $\kappa_\hbar$ by just $\hbar$, for brevity.)

In each of these cases we take $\mathcal{A} \subset \mathfrak{z}(\mathcal{B}_0)$ to be the image of $\mathfrak{z}_{\mathfrak{g},X} := \mathfrak{z}(\mathcal{A}_{\mathfrak{g},\mathrm{crit},X})$. The functoriality of the construction in Sect. 9.2 makes the assertion of Proposition-Construction 9.7 manifest.

9.8. Let us make several additional remarks on the above construction.

First, we see explicitly that the homomorphism $\varphi : \mathfrak{z}_\mathfrak{g} \to \widehat{\mathfrak{H}}_{\mathrm{crit}}$ is indeed a Poisson map, and the action of $N^*_{\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}/\mathfrak{z}_\mathfrak{g}}$ on the category

$$
\left(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_\mathfrak{g}}{\otimes} \mathfrak{z}^{\mathrm{reg}}_\mathfrak{g} \right) -\mathrm{mod} \simeq \mathfrak{H}_{\mathrm{crit},X} -\mathrm{mod}_{\mathrm{reg}}
$$

corresponds to the action of the algebroid $N^*_{\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}/\mathfrak{z}_\mathfrak{g}}$ on the ind-scheme

$$
\mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) \underset{\mathrm{Spec}(\mathfrak{z}_\mathfrak{g})}{\times} \mathrm{Spec}(\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}).
$$

Secondly, assume that $\mathfrak{T}_\hbar$ was a flat family of $\mathfrak{H}_{\hbar,X}$-modules, such that the action of $\mathfrak{z}_\mathfrak{g}$ on $\mathfrak{T}_0$ factors through $\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}$.

On the one hand, by Proposition-Construction 9.4(2), $\mathfrak{T}_{\mathrm{crit}}$ is naturally $N^*_{\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}/\mathfrak{z}_\mathfrak{g}}$-equivariant, as an object of $\left(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_\mathfrak{g}}{\otimes} \mathfrak{z}^{\mathrm{reg}}_\mathfrak{g} \right) -\mathrm{mod}$. Therefore, $\mathbb{W}(\mathfrak{T}_{\mathrm{crit}})$ is $N^*_{\mathfrak{z}^{\mathrm{reg}}_\mathfrak{g}/\mathfrak{z}_\mathfrak{g}}$-equivariant as an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_{\mathrm{reg}}$, and hence carries an action of $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

On other hand, we can consider the family $\mathcal{M}_\hbar := \mathbb{W}(\mathfrak{T}_\hbar)$ as in Sect. 8.2(3), and hence $\mathcal{M}_{\mathrm{crit}} \simeq \mathbb{W}(\mathfrak{T}_{\mathrm{crit}})$ acquires a $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action.

The following assertion results from the definitions:

Lemma 9.9. *Under the above circumstances, the two $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions on $\mathbb{W}(\mathfrak{T}_{\mathrm{crit}})$ coincide.*

9.10.   Let us recall the isomorphism of ind-schemes (4.9)

$$\mathrm{Spec}(\widehat{\mathfrak{H}}_{\mathrm{crit}}) \underset{\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})}{\times} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \simeq \underset{w \in W}{\sqcup} \mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}\,.$$

On the one hand, we have the action of $N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{Z}_{\mathfrak{g}}}$ on the LHS, described in Sect. 9.8. On the other hand, we have a natural action of $\mathsf{isom}_{\check{G}.\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$ on the RHS.

Proposition 9.11. *Under the identification $v_{\mathrm{geom}} : N^*_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{Z}_{\mathfrak{g}}} \simeq \mathsf{isom}_{\check{G}.\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$ the above actions on the two sides of (4.9) coincide.*

Proof. Using Theorem 8.4, we obtain that the assertion of the proposition is equivalent to the following generalization of Lemma 8.11:

The action of $N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}/\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}$ on $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$, resulting from the Poisson structure on $\mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho)$ and the map $\mathrm{MT} : \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}}^\rho) \to \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$, identifies via

$$N^*_{\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}/\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)} \overset{(8.4)}{\simeq} \mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}} \text{ and } \mathrm{Conn}_{\check{H}}(\omega_{\mathcal{D}^\times}^\rho) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} \overset{(4.8)}{\simeq} \underset{w \in W}{\sqcup} \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{th},\mathrm{reg}}$$

with the natural $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$-action on $\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$.

Using Lemma 8.10, this is, in turn, equivalent to the fact that the canonical action of $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}(\mathcal{D}^\times)$ is such that the induced action of $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}}$ on $\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}(\mathcal{D}^\times) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ is compatible with the action of $\mathsf{isom}_{\check{G},\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on $\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}$ under the map

$$\mathrm{MOp}_{\check{\mathfrak{g}},\mathrm{gen}}(\mathcal{D}^\times) \underset{\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}^{\mathrm{reg}} \hookrightarrow \mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{reg}}\,.$$

The required compatibility follows from the construction of the latter map in [FG2], Sect. 3.6.

$\square$

9.12. **The functor G and algebroids.** Let us now return to the setting of Theorem 4.8. Recall that by Sect. 7 for every $w \in W$ there exists a functor $_w\mathsf{Q} : \mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}) \to \mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$ and an isomorphism

$$_w\mathbb{W} \simeq \mathsf{G} \circ {}_w\mathsf{Q}.$$

We claim now that the functor $_w\mathsf{Q}$ is compatible with the action of $\mathsf{isom}_{\check{G}.\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$. Indeed, this follows from the corresponding properties of the functors $_w\mathbb{W}$ and G, established above, and Proposition 7.3.

Recall now that the functor $_w\mathsf{Q}$ has the form

$$\mathcal{T} \mapsto \mathcal{L}_w^{\mathrm{twist}} \underset{\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})}{\otimes} \mathcal{T},$$

for some $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$-bimodule $\mathcal{L}_w^{\mathrm{twist}}$. Moreover, we have shown that $\mathcal{L}_w^{\mathrm{twist}}$, regarded as a right $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$-module, is a line bundle.

Corollary 9.13.

(1) *The left and right actions of $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$ on $\mathcal{L}_w^{\mathrm{twist}}$ coincide.*

(2) *There exists a (unique) $\mathsf{isom}_{\check{G}.\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant structure on $\mathcal{L}_w^{\mathrm{twist}}$, which induces the above $\mathsf{isom}_{\check{G}.\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant structure on $_w\mathsf{Q}$.*

Proof. Let us choose a trivialization of $\mathcal{L}_w^{\mathrm{twist}}$ as a right $\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$-module. Then the left action of $\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ defines a homomorphism $\gamma : \mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}) \to \mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$. The assertion of the first point of the corollary is equivalent to the fact that γ equals the identity map.

However, the fact that the functor $\gamma^* : \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}) \to \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ is compatible with the $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-action on this category implies that γ commutes with the action of this algebroid. In addition, from Theorem 4.10, we know that $\gamma|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$ equals the identity map.

Hence, γ is the identity map, restricted to the minimal $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-stable subscheme of $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$, that contains $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$. But since this minimal subscheme is $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$ itself, assertion (1) of the lemma follows, while assertion (2) is evident.

$\square$

Thus, to finish the proof of Theorem 4.8, it is sufficient to show the following:

Theorem 9.14. *There exists a canonical isomorphism of line bundles*

$$\mathcal{L}_w^{\mathrm{twist}} \simeq \mathcal{L}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})},$$

that respects the $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant structures.

We recall that the line bundle on the RHS is by definition the restriction of the line bundle $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})}$ to $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$; the $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant structure on it comes from the equivariant structure on $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})}$ with respect to the groupoid $\mathsf{Isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$.

9.15. Some corollaries. Let $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ be $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$- equivariant. Then by Proposition-Construction 9.7, the corresponding Wakimoto module $_w\mathbb{W}(\mathcal{T})$ will carry an action of the renormalized algebra $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

Considering $\mathcal{L}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})}{\otimes} \mathcal{T}$ with its $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant structure, from Sect. 9.1 we obtain that $\mathsf{G}\left(\mathcal{L}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})}{\otimes} \mathcal{T}\right)$ also carries an action of $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

Now, the fact that the isomorphism of Theorem 9.14 is $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant implies that the isomorphism

$$_w\mathbb{W}(\mathcal{T}) \simeq \mathsf{G}\left(\mathcal{L}_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})} \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})}{\otimes} \mathcal{T}\right)$$

of Theorem 4.8 respects the $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

As an example let us take $\mathcal{T} = \mathcal{O}_{\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}}$. We obtain:

Corollary 9.16. *The isomorphism*

$$\Gamma^{\mathrm{Hecke}_\mathfrak{Z}}(\mathcal{W}_1^\mathfrak{Z}) \simeq \mathbb{W}_{\mathrm{crit},0}$$

of Theorem 4.11 respects the $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

Let us give a direct proof of this corollary.

Proof. By construction, $\mathcal{W}_1^\mathfrak{Z}$, as an $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}$-equivariant object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_\mathfrak{Z}}$ –mod, is a quotient of the direct sum of objects of the form $\mathrm{Ind}^{\mathrm{Hecke}_\mathfrak{Z}}(j_{\check{\lambda},*,\mathrm{Gr}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}_{\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}}$. Hence, it is enough to show that each of the morphisms

$$\Gamma(\mathrm{Gr}_G, j_{\check{\lambda},\mathrm{Gr}_G,*}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}_{\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}}}^{-\check{\lambda}} \to \mathbb{W}_{\mathrm{crit},0}$$

is compatible with the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

However, the latter morphism is by definition the composition

$$\Gamma(\mathrm{Gr}_G, j_{\check{\lambda}, \mathrm{Gr}_G, *}) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{1,\mathfrak{Z}}} \simeq \left(j_{\check{\lambda}, *} \star \mathbb{V}_{\mathrm{crit}} \right) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{1,\mathfrak{Z}}} \overset{\phi}{\to} \left(j_{\check{\lambda}, *} \star \mathbb{W}_{\mathrm{crit},0} \right) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{1,\mathfrak{Z}}} \simeq$$

$$\simeq \left(\mathbb{W}_{\mathrm{crit},0} \underset{\mathfrak{H}^{\mathrm{reg}}_{\mathrm{crit}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathfrak{H}} \right) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{\mathrm{Fl}^{\check{G}}_{1,\mathfrak{Z}}} \twoheadrightarrow \mathbb{W}_{\mathrm{crit},0},$$

where all the arrows are compatible with the $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions (see Sect. 8.8).

$\square$

Let us return to the general setting of Theorems 4.8 and 9.14, and consider the semi-infinite cohomology functor

$$\mathcal{M} \mapsto H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M} \otimes \Psi_0) : \widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_{\mathrm{reg}} \to \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} -\mathrm{mod}.$$

Recall (see Section 18.3 of [FG2]) that this functor is naturally compatible with the $\mathsf{isom}_{\check{G}, \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$-actions on both categories.

Note, that on the other hand, by Proposition 7.4, the composite functor

$$\mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}) \overset{\mathsf{G}}{\to} \widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}_{\mathrm{reg}} \to \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} -\mathrm{mod}$$

is isomorphic to the identity functor. Moreover, from the construction, it is easy to see that this isomorphism respects the $\mathsf{isom}_{\check{G}, \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$-equivariant structures.

On the other hand, by (7.2), we have a functorial (but dependent on some choices) isomorphism

$$H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], {}_w\mathbb{W}(\mathcal{T}) \otimes \Psi_0) \simeq \mathcal{T}$$

for $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})$.

Thus, combining this with the assertion of Theorem 4.8, we obtain a functorial isomorphism

$$\mathcal{T} \simeq \mathcal{L}^{\check{\rho}-w(\check{\rho})}_{\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}} \underset{\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}})}{\otimes} \mathcal{T},$$

i.e., a trivialization of the line bundle $\mathcal{L}^{\check{\rho}-w(\check{\rho})}_{\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}}$ over $\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}$. Let us explain what this trivialization is.

First, let us observe that the choice of a character Ψ_0 amounts to a trivialization of the H-torsor $\omega^{\check{\rho}}_x$. However, since different choices of Ψ_0 are $H[[t]]$-conjugate, we obtain that when we apply the functor $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$ to $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules that are I-equivariant, the spaces that we obtain for different choices of Ψ_0 are canonically isomorphic. In other words, on the category $\widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}^I$, the functor $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$ is well-defined.

For the same reason, if $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}} -\mathrm{mod}^I$ is in addition equivariant with respect to $\mathrm{Aut}(\mathcal{D})$, then the cohomology $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$ acquires a $\mathrm{Aut}(\mathcal{D})$-action.

Canonically, we have an isomorphism

(9.3) $$H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], {}_w\mathbb{W}(\mathcal{T}) \otimes \Psi_0) \simeq \mathcal{T} \otimes \omega^{\langle w(\rho)-\rho, \check{\rho}\rangle}_x.$$

Thus, rather than trivializing the line bundle $\mathcal{L}^{\check{\rho}-w(\check{\rho})}_{\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}}$ we need to identify it with $\mathrm{Fun}(\mathrm{Fl}^{\check{G}}_{w,\mathrm{th},\mathfrak{Z}}) \otimes \omega^{\langle w(\rho)-\rho, \check{\rho}\rangle}_x$ in a $\mathrm{Aut}(\mathcal{D})$-equivariant way. It is easy to see that there exists a unique such identification, with induces the isomorphism (6.4) over $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}$.

9.17. **Proof of Theorem 9.14.** Recall the algebroid $\mathsf{isom}^{\mathrm{nilp}}_{\mathrm{Op}_{\check{\mathfrak{g}}}}$ over the scheme $\mathrm{Op}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$ (see [FG2], Sect. 4.5), and let $\mathsf{isom}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}$ denote the corresponding algebroid over $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}})$.

By *loc. cit.*, it preserves the subscheme $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}) \subset \mathrm{Spec}(\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}})$. Hence, the restriction $\mathsf{isom}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}|_{\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})}$ has a natural structure of Lie algebroid; we will denote it by $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$. By [FG2], Sect. 4.5, $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$ identifies with the sub-algebroid of $\mathsf{isom}_{\check{G},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$, corresponding to the reduction of $\mathcal{P}_{\check{G},\mathfrak{Z}}$ to $\check{B}$.

Since $\check{B}$ acts transitively along the fibers of $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}} \to \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, and since the stabilizers of any point of $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}$ in $\mathsf{isom}_{\check{G},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$ and $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$ coincide, it suffices to show that there exists a $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$-equivariant isomorphism

$$(9.4) \qquad \mathcal{L}^{\mathrm{twist}}_w\big|_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}} \simeq \mathcal{L}^{\check{\rho}-w(\check{\rho})}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}$$

of line bundles on $\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}$.

Recall that along with the renormalized algebra $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, there exists its version $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, corresponding to the quotient $\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}$ of $\mathfrak{Z}_{\mathfrak{g}}$ (see [FG2], Sect. 7.4).

As in Proposition-Construction 9.7, we obtain that if

$$\mathcal{T} \in \mathrm{QCoh}\Big(\mathrm{Spec}\big(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}\big)\Big),$$

is $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}$-equivariant, then the Wakimoto module $\mathbb{W}(\mathcal{T})$ acquires an action of the renormalized algebra $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

Similarly, if $\mathcal{T} \in \mathrm{QCoh}(\mathrm{Fl}^{\check{G}}_{\mathfrak{Z}})$ is $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$-equivariant, then $\mathsf{G}(\mathcal{T})$ also acquires an action of $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}}) \underset{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$.

Recall now the isomorphism

$$(9.5) \qquad \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{W}^{\mathfrak{Z}}_w) \simeq \mathbb{W}_{\mathrm{crit},w(\rho)-\rho,\mathrm{reg}} \otimes \omega^{\langle \rho-w(\rho),\check{\rho}\rangle}_x \overset{(6.4)}{\simeq} \mathbb{W}\big(\mathcal{L}^{w(\check{\rho})-\check{\rho}}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}\big)$$

given by Theorem 4.10. By the definition of $\mathcal{L}^{\mathrm{twist}}_w$, we obtain an isomorphism of line bundles appearing in (9.4). The fact that this isomorphism respects the $\mathsf{isom}_{\check{B},\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}$-action is equivalent to the fact that the isomorphism of (9.5) respects the action of $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

This is, in turn, equivalent to the map

$$\mathbb{V}_{\mathrm{crit}} \to \mathbb{W}\big(\mathcal{L}^{w(\check{\rho})-\check{\rho}}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}\big),$$

of (6.6) being compatible with the $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action. Recall that the latter morphism comes by adjunction from a map

$$(9.6) \qquad \mathbb{V}_{\mathrm{crit}} \to \mathrm{Av}_{G[[t]]/I}\Big(\mathbb{W}\big(\mathcal{L}^{w(\check{\rho})-\check{\rho}}_{\mathrm{Fl}^{\check{G}}_{w,\mathfrak{Z}}}\big)\Big).$$

Thus, we need to show that the map in (9.6) is compatible with the action of $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, where on the RHS it is defined by functoriality as in Sect. 8.2(**4**). The latter essentially follows from the definition of the map (6.6):

Consider the family of Wakimoto modules $\mathbb{W}_{\hbar,w(\rho)-\rho}$, which is, by definition, induced from the $\mathfrak{H}_{\hbar,X}$-module

$$\pi_{\hbar,w_0(w(\rho)-\rho)} := \mathrm{Ind}^{\widehat{\mathfrak{h}}_{\hbar}}_{\mathfrak{h}[[t]]}(\mathbb{C}^{w_0(w(\rho)-\rho)}).$$

Its fiber at the critical level

$$\pi_{\mathrm{crit},w_0(w(\rho)-\rho)} \simeq \mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}$$

is supported over $\mathrm{Spec}\big(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes} \mathfrak{z}_{\mathfrak{g}}^{\mathrm{nilp}}\big)$, hence, it acquires an action of $\mathsf{isom}_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{nilp}}}$. By the nilp-version of Proposition-Construction 9.7, the module $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}$ is endowed an action of $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, and by Lemma 9.9, this action coincides with the one coming from the family $\mathbb{W}_{\hbar,w(\rho)-\rho}$.

We have an $\hbar$-family of maps

$$\mathbb{V}_{\hbar} \to h^{\ell(w)}\Big(\mathrm{Av}_{G[[t]]/I}\big(\mathbb{W}_{\hbar,w(\rho)-\rho}\big)\Big)$$

and, hence, the correspoding map at the critical level

(9.7) $$\mathbb{V}_{\mathrm{crit}} \to h^{\ell(w)}\Big(\mathrm{Av}_{G[[t]]/I}\big(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}\big)\Big)$$

is compatible with the $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action. A compatibility of constructions in points **3** and **4** of Sect. 8.2 shows that the $U^{\mathrm{ren,nilp}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-action on the RHS of (9.7) coincides with the one coming by functoriality from its action on $\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}$ via Proposition-Construction 9.7.

Let $\imath_w$ denote the closed embedding

$$\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}} \hookrightarrow \mathrm{Spec}(\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}).$$

We have a canonical $\mathsf{isom}_{\check{B},\mathfrak{z}_{\mathfrak{g}}^{\mathrm{nilp}}}$-equivariant isomorphism:

$$\mathrm{Av}_{G[[t]]/I}\Big(\mathbb{W}\big(\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}}^{w(\check{\rho})-\check{\rho}}\big)\Big) \simeq \mathrm{Av}_{G[[t]]/I}\big(\mathbb{W}_{\mathrm{crit},w(\rho)-\rho}\big) \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}{\overset{L}{\otimes}} \imath_{w*}\Big(\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}}^{w(\check{\rho})-\check{\rho}}\Big).$$

The map (9.6) is, by construction, obtained from the map (9.7) and the identification

$$L^{\ell(w)}\imath_w^*\Big(\imath_{w*}\big(\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}})\big)\Big) \simeq \mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{z}}^{\check{G}}}^{\check{\rho}-w(\check{\rho})}.$$

Thus, we need to see that the latter is compatible with the $\mathsf{isom}_{\check{B},\mathfrak{z}_{\mathfrak{g}}^{\mathrm{nilp}}}$-actions.

Translating it using the isomorphism $\mathrm{map}_{\mathrm{geom}}^{M}$, we obtain that the LHS of the above expression identifies with the line bundle

$$\Lambda^{\ell(w)}\Big(N^*_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}} / \mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{nilp}}}\Big)$$

over $\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}$. By Theorem 3.6.2 of [FG2], the latter does indeed identify with $\mathcal{L}_{\mathrm{MOp}_{\check{\mathfrak{g}}}^{w,\mathrm{reg}}}^{\check{\rho}-w(\check{\rho})}$ in a $\mathsf{isom}_{\mathrm{Op}_{\check{\mathfrak{g}}}}^{\mathrm{nilp}}$-equivariant way, as required.

10. Renormalized Wakimoto modules

In this section we will study a particular family of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod-modules, obtained by a certain renormalization procedure. We will show that they coincide with Wakimoto modules corresponding to some particular quasi-coherent sheaves on the formal neighborhoods of the Schubert strata in $\mathrm{Fl}_{\mathfrak{z}}^{\check{G}}$.

10.1. Recall the context of Sect. 9.2. Let $\mathbf{B}_\hbar$–mod be the category of all (discrete) $\mathbf{B}_\hbar$-modules, and let $\mathbf{B}_\hbar$–mod$^{fl} \subset \mathbf{B}_\hbar$–mod be the full subcategory consisting of $\mathbb{C}[\hbar]$-flat modules. Let

(10.1) $$\mathbf{B}_\hbar\text{–mod}^{fl,\mathrm{reg}} \subset \mathbf{B}_\hbar\text{–mod}^{fl}$$

be the full subcategory, consisting of modules $\mathcal{N}_\hbar$, for which the action of $\mathbf{A}$ on $\mathcal{N}_0 := \mathcal{N}_\hbar/\hbar \cdot \mathcal{N}_\hbar$ factors through $\mathbf{A}^{\mathrm{reg}}$. (Recall that in this case $\mathcal{N}_0$ acquires a natural $\mathbf{B}^{\mathrm{ren}}$-action.)

Proposition-Construction 10.2. *The tautological embedding* (10.1) *admits a left adjoint.*

Proof. For $\mathcal{M}_\hbar \in \mathbf{B}_\hbar$–mod^{fl} let $\mathcal{M}_\hbar^\sharp$ denote its modification spanned by the symbols $\frac{m_\hbar}{\hbar}$ where $m_\hbar$ is such that $m_0 \in \mathbf{I} \cdot \mathcal{M}_0$. We have a canonical map $\mathcal{M}_\hbar \to \mathcal{M}_\hbar^\sharp$ and a short exact sequence

$$0 \to \mathcal{M}_0/\mathbf{I} \cdot \mathcal{M}_0 \to \mathcal{M}_0^\sharp \to \mathbf{I} \cdot \mathcal{M}_0 \to 0.$$

Let us denote by $\mathcal{M}_\hbar \mapsto \mathcal{M}_\hbar^{k \cdot \sharp}$ the k-th iteration of the functor $\mathcal{M}_\hbar \mapsto \mathcal{M}_\hbar^\sharp$. The fiber $\mathcal{M}_0^{k \cdot \sharp}$ admits a $k + 1$-term filtration $\left(\mathcal{M}_0^{k \cdot \sharp}\right)_i$, such that

$$\begin{cases} \mathrm{gr}^j\left(\mathcal{M}_0^{k \cdot \sharp}\right) \simeq \mathbf{I}^{j-1} \cdot \mathcal{M}_0/\mathbf{I}^j \cdot \mathcal{M}_0, \ 1 \le j \le k \text{ and} \\ \mathrm{gr}^{k+1}\left(\mathcal{M}_0^{k \cdot \sharp}\right) \simeq \mathbf{I}^k \cdot \mathcal{M}_0. \end{cases}$$

Moreover, the submodule $\left(\mathcal{M}_0^{k \cdot \sharp}\right)_k \subset \mathcal{M}_0^{k \cdot \sharp}$ is annihilated by $\mathbf{I}$.

Thus, we obtain a sequence of maps

$$... \to \mathcal{M}_\hbar^{(k-1) \cdot \sharp} \to \mathcal{M}_\hbar^{k \cdot \sharp} \to \mathcal{M}_\hbar^{(k+1) \cdot \sharp} \to ...$$

and we set

$$\mathcal{M}_\hbar^{\mathrm{ren}} := \varinjlim_k \mathcal{M}_\hbar^{k \cdot \sharp}.$$

The above computation of fibers of $\mathcal{M}_\hbar^{k \cdot \sharp}$ implies that $\mathcal{M}_\hbar^{\mathrm{ren}}$ belongs to $\mathbf{B}_\hbar$–mod$^{\mathrm{reg},fl}$. It satisfies the required adjunction property by construction.

$\square$

For $\mathcal{M}_\hbar$ as above, let $\mathcal{M}_0^{\mathrm{ren}}$ denote the $\mathbf{B}^{\mathrm{reg}}$-module $\mathcal{M}_\hbar^{\mathrm{ren}}/\hbar \cdot \mathcal{M}_\hbar$. By construction, it carries an action of B_0^{ren}. As a $\mathbf{B}^{\mathrm{reg}}$-module it is equipped with an increasing filtration, labeled by positive integers, with

(10.2) $$\mathrm{gr}^j(\mathcal{M}_0^{\mathrm{ren}}) \simeq \mathbf{I}^j \cdot \mathcal{M}_0/\mathbf{I}^{j+1} \cdot \mathcal{M}_0.$$

10.3. Let us consider the family of chiral algebras $\mathfrak{H}_{\hbar,X}$; let $\widehat{\mathfrak{h}}_\hbar$ be the central extension of $\mathfrak{h}((t))$, corresponding to the point $x \in X$, and let $\mathbf{B}_\hbar := \widehat{\mathfrak{H}}_\hbar$ be the corresponding family of associative topological algebras. As in Sect. 9.6, we let $\mathbf{I} \subset \mathbf{B}_0 := \widehat{\mathfrak{H}}_{\mathrm{crit}}$ be the image of $\ker(\mathfrak{Z}_\mathfrak{g} \to \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}})$ under the map $\varphi : \mathfrak{Z}_\mathfrak{g} \to \widehat{\mathfrak{H}}_{\mathrm{crit}}$.

For $w \in W$ let $\pi_{\hbar,w}$ denote the $\widehat{\mathfrak{H}}_\hbar$-module $\mathrm{Ind}_{\mathfrak{h}[[t]]}^{\widehat{\mathfrak{h}}_\hbar}(\mathbb{C}^{w_0(w(\rho)-\rho)})$. We let $\pi_{\mathrm{crit},w}$ denote the fiber of $\pi_{\hbar,w}$ at the critical level. Let $\pi_{\hbar,w}^{\mathrm{ren}}$ be the $\widehat{\mathfrak{H}}_\hbar$-module, corresponding to $\pi_{\hbar,w}$ via Proposition-Construction 10.2.

Let $\pi_{\mathrm{crit},w}^{\mathrm{ren}}$ denote the fiber of $\pi_{\hbar,w}^{\mathrm{ren}}$ at the critical level. By Proposition-Construction 9.4, $\pi_{\mathrm{crit},w}^{\mathrm{ren}}$ is equivariant with respect to the algebroid $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^* \simeq \mathsf{isom}_{\check{G}, \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}}$, which amounts to an action of $N_{\mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}/\mathfrak{Z}_\mathfrak{g}}^*$ on $\pi_{\mathrm{crit},w}^{\mathrm{ren}}$ compatible with its action on $\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{Z}_\mathfrak{g}}{\otimes} \mathfrak{Z}_\mathfrak{g}^{\mathrm{reg}}$.

The main result of this section is the following explicit description of $\pi_{\mathrm{crit},w}^{\mathrm{ren}}$:

Recall that Dist_w denotes the quasi-coherent sheaf on $\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}$, underlying the left D-module of distributions on the subscheme $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$. Consider the object

$$\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{w(\check{\rho})-\check{\rho}} \underset{\mathcal{O}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}}{\otimes} \mathrm{Dist}_w \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}),$$

which we think of as a $(\widehat{\mathfrak{H}}_{\mathrm{crit}} \underset{\mathfrak{Z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$-module via the identification (4.9). It is naturally equivariant with respect to $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}} \simeq N_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{Z}_{\mathfrak{g}}}^*$.

Theorem 10.4. *There exists a canonical isomorphism*

$$\pi_{\mathrm{crit},w}^{\mathrm{ren}} \simeq \mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{w(\check{\rho})-\check{\rho}} \underset{\mathcal{O}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}}{\otimes} \mathrm{Dist}_w \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}),$$

compatible with the $N_{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}/\mathfrak{Z}_{\mathfrak{g}}}^*$-action.

10.5. **Application to Wakimoto modules and BGG type resolution.** Let $\mathbb{W}_{\hbar,w}$ be the $\hbar$-family of Wakimoto modules, induced from $\pi_{\hbar,w}$. Applying the above renormalization construction to $\mathbf{B}_\hbar = \widehat{\mathcal{A}}_{\mathfrak{g},\hbar,x}$ and the module $\mathbb{W}_{\hbar,w}$ with respect to the ideal $\ker(\mathfrak{Z}_{\mathfrak{g}} \to \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ we obtain the renormalized family of Wakimoto modules, denoted $\mathbb{W}_{\hbar,w}^{\mathrm{ren}}$.

Let $\mathbb{W}_{\mathrm{crit},w}^{\mathrm{ren}}$ denote the fiber of $\mathbb{W}_{\hbar,w}^{\mathrm{ren}}$ at the critical level. As in Sect. 9.6 we have:

Lemma 10.6. *The lattices $\mathbb{W}_{\hbar,w}^{\mathrm{ren}}$ and $\mathbb{W}_\hbar(\pi_{\hbar,w}^{\mathrm{ren}})$ in the localization of $\mathbb{W}_{\hbar,w}$ with respect to $\hbar$ coincide.*

Hence, in particular, $\mathbb{W}_{\mathrm{crit},w}^{\mathrm{ren}}$ is isomorphic to $\mathbb{W}_{\mathrm{crit}}(\pi_{\mathrm{crit},w}^{\mathrm{ren}})$. Moreover, by Lemma 9.9, the above isomorphism respects the $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-actions.

Combining this with Theorem 4.8, we obtain an isomorphism of $U^{\mathrm{ren},\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$-modules

$$(10.3) \qquad\qquad \mathbb{W}_{\mathrm{crit},w}^{\mathrm{ren}} \simeq \mathsf{G}(\mathrm{Dist}_w).$$

Combining Theorem 10.4 with Corollary 4.13 we obtain a right resolution of $\mathbb{V}_{\mathrm{crit}}$:

$$\mathbb{V}_{\mathrm{crit}} \to C^0 \xrightarrow{\delta^0} C^1 \to ...,$$

whose k-th term is

$$\underset{w \in W, \ell(w)=k}{\oplus} \mathbb{W}_{\mathrm{crit},w}^{\mathrm{ren}}.$$

Note that for $w = 1$ the module Dist_w is just $\mathrm{Fun}(\mathrm{Fl}_{1,\mathfrak{Z}}^{\check{G}})$ and the corresponding Wakimoto module is $\mathbb{W}_{\mathrm{crit},0}$. In the case when w is a simple reflection, the modules Dist_w, and the corresponding Wakimoto modules were constructed in [FF3], and the differential δ^0 was described there explicitly as the sum of certain degenerations of the "screening operators" at the critical level. It was conjectured in [FF3] that this complex may be extended to a resolution of $\mathbb{V}_{\mathrm{crit}}$, which is a particular degeneration of the BGG type resolution of $\mathbb{V}_{\mathrm{crit}}$ for generic levels.

Thus, we have obtained a proof of this conjecture. However, it would be interesting to find explicit formulas for the higher differentials $\delta^k : C^k \to C^{k+1}$ of this resolution in terms of the screening operators.

10.7.   The rest of this section is devoted to the proof of Theorem 10.4. Observe that the pullback $\mathsf{isom}_{\check{G},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}|_{\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}}$ has a natural structure of algebroid on $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$; we will denote it by $\mathsf{isom}_{\check{G},w,\mathrm{th}}$. Let $\mathsf{isom}_{\check{G},w} \subset \mathsf{isom}_{\check{G},w,\mathrm{th}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$ be the corresponding algebroid on $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$ it contains $\mathsf{isom}_{\check{B},w} := \mathsf{isom}_{\check{B},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$ as a sub-algebroid.

Since the action of $\mathsf{isom}_{\check{G},w,\mathrm{th}}$ on $\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$ is transitive, from Kashiwara's theorem (see, e.g., [FG1], Sect. 7) we obtain the following:

Lemma 10.8. *Let $\mathfrak{T} \in \mathrm{QCoh}(\mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}})$ be a module over $\mathsf{isom}_{\check{G},w,\mathrm{th}}$. Assume that the $\mathcal{O}$-submodule $\mathfrak{T}^0 \subset \mathfrak{T}$ of sections supported over $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}$ is isomorphic as a $\mathsf{isom}_{\check{B},w}$-module to $\mathcal{L}_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}^{\check{\lambda}}$ for some $\check{\lambda} \in \check{\Lambda}$. Then $\mathfrak{T}$ is isomorphic to $\mathcal{L}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}^{w(\check{\rho})-\check{\rho}+\check{\lambda}} \underset{\mathcal{O}_{\mathrm{Fl}_{\mathfrak{Z}}^{\check{G}}}}{\otimes} \mathrm{Dist}_w$ as a $\mathsf{isom}_{\check{G},w,\mathrm{th}}$-module.*

We claim that the conditions of this lemma are satisfied for $\mathfrak{T} = \pi_{\mathrm{crit},w}^{\mathrm{ren}}$ and $\check{\lambda} = 0$. Consider the canonical filtration on $\pi_{\mathrm{crit},w}^{\mathrm{ren}}$. For every k we have a map

$$\left(\mathsf{isom}_{\check{G},w,\mathrm{th}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}/\mathsf{isom}_{\check{G},w}\right) \underset{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})}{\otimes} \left(\left(\pi_{\mathrm{crit},w}^{\mathrm{ren}}\right)_k/\left(\pi_{\mathrm{crit},w}^{\mathrm{ren}}\right)_{k-1}\right) \to$$

$$\to \left(\pi_{\mathrm{crit},w}^{\mathrm{ren}}\right)_{k+1}/\left(\pi_{\mathrm{crit},w}^{\mathrm{ren}}\right)_k,$$

which by (10.2) induces an isomorphism

$$\mathrm{gr}^k\left(\pi_{\mathrm{crit},w}^{\mathrm{ren}}\right) \simeq \pi_{\mathrm{crit},w} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}{\otimes} \mathrm{Sym}_{\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})}^k\left(\mathsf{isom}_{\check{G},w,\mathrm{th}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}/\mathsf{isom}_{\check{G},w}\right).$$

In particular, we obtain that $\pi_{\mathrm{crit},w} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}) \subset \pi_{\mathrm{crit},w}^{\mathrm{ren}}$ equals the subspace consisting of sections supported scheme-theoretically on $\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}} \subset \mathrm{Fl}_{w,\mathrm{th},\mathfrak{Z}}^{\check{G}}$.

Hence, it remains to see that

$$\pi_{\mathrm{crit},w} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \simeq \pi_{\mathrm{crit},w} \underset{\mathfrak{H}_{\mathrm{crit}}^{\mathrm{RS},w_0(w(\rho)-\rho)}}{\otimes} \mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$$

is isomorphic to $\mathrm{Fun}(\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}})$, as a module over $\mathsf{isom}_{\check{B},w} \simeq \mathsf{isom}_{\check{B},\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}}|_{\mathrm{Fl}_{w,\mathfrak{Z}}^{\check{G}}}$. The latter fact follows from the nilp-version of Proposition 9.11.

References

[ABG]   S. Arkhipov, R. Bezrukavnikov and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, Journal of AMS **17** (2004) 595–678.

[ABBGM]   S. Arkhipov, R. Bezrukavnikov, A. Braverman, D. Gaitsgory and I. Mirković, *Modules over the small quantum group and semi-infinite flag manifold*, math.AG/0505280, to appear in Transformation Groups.

[AG]   S. Arkhipov and D. Gaitsgory, *Another realization of the category of modules over the small quantum group*, Adv. Math. **173** (2003) 114–143.

[BD]   A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at http://www.math.uchicago.edu/~arinkin/langlands/

[CHA]   A. Beilinson and V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications **51**, AMS, 2004.

[FF1]   B. Feigin and E. Frenkel, *A family of representations of affine Lie algebras*, Russ. Math. Surv. **43** (1988) no. 5, 221–222.

[FF2]   B. Feigin and E. Frenkel, *Affine Kac-Moody Algebras and semi-infinite flag manifolds*, Comm. Math. Phys. **128** (1990) 161–189.

[FF3]   B. Feigin and E. Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras*, in *Infinite Analysis*, eds. A. Tsuchiya, T. Eguchi, M. Jimbo, Adv. Ser. in Math. Phys. **16**, 197–215, Singapore: World Scientific, 1992.

[F]   E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005) 297–404.

[FB]   E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs **88**, Second Edition, AMS, 2004.

[FG1]   E. Frenkel and D. Gaitsgory, *D-modules on the affine Grassmannian and representations of affine Kac-Moody algebras*, Duke Math. J. **125** (2004) 279–327.

[FG2]   E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, Preprint math.RT/0508382.

[FG3]   E. Frenkel and D. Gaitsgory, *Fusion and convolution: applications to affine Kac-Moody algebras at the critical level*, Preprint math.RT/0511284.

[FG4]   E. Frenkel and D. Gaitsgory, *Localization of $\widehat{\mathfrak{g}}$-modules on the affine Grassmannian* Preprint math.RT/0512562.

[Ga]   D. Gaitsgory, *The notion of category over an algebraic stack*, math.AG/0507192.

[MV]   I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Preprint math.RT/0401222.

[W]   M. Wakimoto, *Fock representations of affine Lie algebra $A_1^{(1)}$*, Comm. Math. Phys. **104** (1986) 605–609.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA
E-mail address: frenkel@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA
E-mail address: gaitsgory@math.harvard.edu